Published by Institute of Physics Publishing for SISSA/ISAS

Received: *December 20, 2002*
Revised: *April 7, 2003*
Accepted: *April 23, 2003*

# The geometry of $D = 11$ Killing spinors

**Jerome P. Gauntlett and Stathis Pakis**

*Department of Physics, Queen Mary, University of London*
*Mile End Rd, London E1 4NS, UK*
*E-mail:* j.p.gauntlett@qmul.ac.uk, s.pakis@qmul.ac.uk

Abstract: We propose a way to classify the local form of all bosonic supersymmetric configurations of $D = 11$ supergravity, using the differential forms that can be constructed as bi-linears from the Killing spinors. We show that the most general bosonic geometries either have a privileged SU(5) or a $(\mathrm{Spin}(7) \ltimes \mathbb{R}^8) \times \mathbb{R}$ structure, depending on whether the Killing vector constructed from the Killing spinor is timelike or null, respectively. In the time-like case we derive the general local form of the geometry and show that it is almost completely determined by a certain SU(5) structure on the ten-dimensional space orthogonal to the orbits of the Killing vector. We also deduce what further conditions must be imposed in order that the equations of motion are satisfied. We illustrate the formalism with some known solutions and also present some new solutions including a rotating generalisation of the resolved membrane solutions and generalisations of the recently constructed $D = 11$ Gödel solution. We also prove some general vanishing theorems for compactifications with flux.

Keywords: M-Theory, Supergravity Models, Differential and Algebraic Geometry.

## Contents



JHEP04(2003)039

## 1. Introduction

Supersymmetric solutions of supergravity theories have played a prominent role in many developments in string theory and it would be useful to have a systematic classification of all such solutions. When the fluxes are all set to zero we know that the supersymmetric geometries must admit covariantly constant spinors and hence must admit metrics with special holonomy. In the riemannian case the possible special holonomy groups that can arise are completely classified. For a relevant discussion on the lorentzian case, see [1, 2].

There are many results in the literature concerning special cases when the fluxes are non-vanishing but they are typically based on special ansatzë[1] and a global picture has been lacking. Here we shall propose a way to classify the local forms of all supersymmetric solutions of $D = 11$ supergravity, independent of ansatz, building on the work of [4]–[9] using the $G$-structures defined by Killing spinors.[2] Moreover, it is clear how to extend the ideas to any supergravity theory. Indeed a complete analysis for $D = 5$ minimal supergravity has already been carried out in [9] (for earlier work on the simpler case of $N = 2$ supergravity in $D = 4$, using techniques specific to $D = 4$, see [12]).

We should mention at the outset, that the classification we are advocating would still leave the very challenging task of determining all of the explicit supersymmetric solutions that can arise within the classes we will discuss. For example, in the special case when the flux is zero, this corresponds to explicitly classifying all special holonomy manifolds, which seems to require fundamentally new mathematical ideas in order to make progress.

We start in section 2 by deriving a number of necessary conditions for a bosonic geometry, consisting of a metric and a four-form, to admit Killing spinors. We first construct differential forms of rank $0, \ldots, 5$ from bi-linears of the Killing spinors. Fierz identities lead to a number of algebraic conditions that these forms must satisfy, while the Killing spinor equation gives a number of differential constraints. For example, the vector fields dual to the one-forms $K$ constructed from the Killing spinors are always Killing. When one of the $K$ is timelike, we show that some of the differential conditions are those of generalised calibrations [13]–[15] for membranes and also for fivebranes, with a small extension for the latter case. The same differential conditions hold when $K$ is null or, when there is more than one Killing spinor, spacelike, which suggests an interesting extension of the notion of generalised calibration.

In section 3, we argue that $G$-structures are very useful for interpreting and organising the results of section 2. We begin by recalling the notion of $G$-structures and their classification and then discuss how they can provide the basis for a classification of all supersymmetric solutions. One important result is that any supersymmetric solution, i.e. preserving at least one Killing spinor (1/32 supersymmetry), will either have a privileged local SU(5) or an $(\mathrm{Spin}(7) \ltimes \mathbb{R}^8) \times \mathbb{R}$ structure that is constructed from the Killing spinor. The two cases are distinguished by whether the Killing vector is time-like or null, respectively.

To be more precise, the Killing vector is either null everywhere or it isn't. The former "null"-case admits a globally[3] defined $(\mathrm{Spin}(7) \ltimes \mathbb{R}^8) \times \mathbb{R}$ structure. In the latter "time-like"-case there is at least a point and hence a neighbourhood where $K$ is timelike. Since the neighbourhood is topologically trivial, the frame bundle can always be trivialised. However, for our purposes it will be important to note that the Killing spinor defines a privileged $D = 11$ SU(5) structure which satisfies certain differential conditions and these

[1]An exception is the recent classification of maximally supersymmetric solutions of $D = 10, 11$ supergravity [3].

[2]For other work relating $G$-structures to supergravity solutions with non-vanishing fluxes, see [10, 11].

[3]We will ignore issues to do with precisely what kinds of singularities we want to allow in physically relevant solutions.

are non-trivial even in the topologically trivial neighbourhood. It should also be noted that the Killing spinor defines a global SU(5) structure if we restrict to regions in which the Killing vector is time-like, which could in fact be the whole manifold, if there are no points in which the Killing vector becomes null.

The time-like case is analysed in detail in section 4. Working in a neighbourhood where $K$ is time-like, we use the Killing spinor equation and the SU(5) structure to determine the general local form of the geometry. Since $K$ is Killing the SU(5) structure in eleven-dimensions turns out to be mostly specified by an SU(5) structure in the ten-dimension base space orthogonal to the orbits of the Killing vector. The only restriction on the ten-dimensional SU(5) structure is that the class $\mathcal{W}_5$, defined later as the Lee-form of the (5,0)-form, is exact and related to the norm of the Killing vector. We find that, much, but not all, of the form of the geometry is determined by the SU(5) structure. In particular, there is a component of the four-form field strength which is undetermined. The reason for this fact is simply that this component drops out of the Killing spinor equation. The necessary and sufficient form of the geometry is presented in (4.20), (4.21) and (4.22). By analysing integrability conditions for the Killing spinor equation, we also determine the extra constraints imposed on those geometries admitting Killing spinors in order that they solve the equations of motion. The extra constraints are presented in (4.29) and (4.30). The equations of motion constrain the component of the four-form that is not determined by supersymmetry alone. Our results allow us to obtain some vanishing theorems for compactifications with flux (for other such theorems in D=10 supergravity with NS three-form flux only, and assuming a restricted class of configurations, see [16, 17, 8]).

The results for the time-like case presented here, and the analogous results for the null-case, which we leave to future work, would classify the local form of the most general bosonic supersymmetric solutions of $D = 11$ supergravity. A refinement of this classification would be to find the additional restrictions placed on the geometries in order that they preserve more than one supersymmetry, and we discuss how in principle this could be done.

In section 5 we use the results for the timelike case to obtain some some new solutions. In [18, 19] (see also [20]–[22]) it was shown that the membrane solution with a transverse manifold of SU(4) holonomy can be resolved by switching on additional four-form flux via a harmonic four-form. Here we will show that one can extend these solutions to include rotation. In [9] a D=5 generalisation of the Gödel solution was constructed. It was shown that it can be uplifted to $D = 11$ where it then preserves 5/8 supersymmetry. The topology of the space is $\mathbb{R}^{11}$ and there is a rotational one-form that lives in an $\mathbb{R}^4$ factor. We will show that there are further solutions with more complicated rotation one-forms.

Section 6 briefly concludes.

## 2. Killing spinors and differential forms

The bosonic fields of $D = 11$ supergravity consist of a metric, $g$, and a three-form potential $A$ with four-form field strength $F = dA$. The action for the bosonic fields is given by

$$S = \frac{1}{2\kappa^2} \int d^{11}x\sqrt{-g}R - \frac{1}{2}F \wedge *F - \frac{1}{6}C \wedge F \wedge F\,, \tag{2.1}$$

where $F = dC$. The equations of motion are thus given by

$$
\begin{aligned}
R_{\mu\nu} - \frac{1}{12}(F_{\mu\sigma_1\sigma_2\sigma_3}F_\nu{}^{\sigma_1\sigma_2\sigma_3} - \frac{1}{12}g_{\mu\nu}F^2) &= 0 \\
d * F + \frac{1}{2}F \wedge F &= 0 .
\end{aligned} \tag{2.2}
$$

We are interested in bosonic solutions to the equations of motion that preserve at least one supersymmetry i.e. solutions that admit at least one Killing spinor, $\epsilon$, which solves

$$
\nabla_\mu \epsilon + \frac{1}{288}[\Gamma_\mu{}^{\nu_1\nu_2\nu_3\nu_4} - 8\delta_\mu^{\nu_1}\Gamma^{\nu_2\nu_3\nu_4}]F_{\nu_1\nu_2\nu_3\nu_4}\epsilon = 0 . \tag{2.3}
$$

Note that due to the presence of the terms involving the four-form the supercovariant derivative appearing in (2.3) takes values in the Clifford algebra and not just the spin subalgebra. Our conventions are outlined in appendix A.

Note that in M-theory, the field-equation for the four-form receives higher order gravitational corrections [23, 24]:

$$
d * F + \frac{1}{2}F \wedge F = -\beta X_8 , \tag{2.4}
$$

where

$$
X_8 = \frac{1}{192}(p_1^2 - 4p_2) , \tag{2.5}
$$

$\beta = 2\pi/T_5$ where $T_5 = (2\pi)^{1/3}/(2\kappa^2)^{2/3}$ is the tension of a single fivebrane, and the Pontryagin-forms are given by

$$
\begin{aligned}
p_1 &= -\frac{1}{8\pi^2}\,\mathrm{tr}\, R^2 \\
p_2 &= -\frac{1}{64\pi^4}\,\mathrm{tr}\, R^4 + \frac{1}{128\pi^4}(\mathrm{tr}\, R^2)^2 .
\end{aligned} \tag{2.6}
$$

Since most of our analysis concerns the Killing spinor equation (2.3) this correction will not play a large role in the following.

Consider a geometry that admits $N$ Killing spinors $\epsilon^i$, $i = 1, \ldots, N$. We can define the following one, two and five-forms that are symmetric in $i, j$:

$$
\begin{aligned}
K_\mu^{ij} &= \bar{\epsilon}^i \Gamma_\mu \epsilon^j \\
\Omega_{\mu_1\mu_2}^{ij} &= \bar{\epsilon}^i \Gamma_{\mu_1\mu_2} \epsilon^j \\
\Sigma_{\mu_1\mu_2\mu_3\mu_4\mu_5}^{ij} &= \bar{\epsilon}^i \Gamma_{\mu_1\mu_2\mu_3\mu_4\mu_5} \epsilon^j .
\end{aligned} \tag{2.7}
$$

We can also define zero, three and four-forms which are anti-symmetric in $i, j$

$$
\begin{aligned}
X^{ij} &= \bar{\epsilon}^i \epsilon^j \\
Y_{\mu_1\mu_2\mu_3}^{ij} &= \bar{\epsilon}^i \Gamma_{\mu_1\mu_2\mu_3} \epsilon^j \\
Z_{\mu_1\mu_2\mu_3\mu_4}^{ij} &= \bar{\epsilon}^i \Gamma_{\mu_1\mu_2\mu_3\mu_4} \epsilon^j .
\end{aligned} \tag{2.8}
$$

Note that that the diagonal forms with $i = j$ are non-vanishing if and only if $\epsilon$ is. Indeed, in our conventions the charge conjugation matrix is given by $C = \Gamma_0$, so the zeroth component of the vector $K$ in an orthonormal frame is given by $K_0^{ii} = -(\epsilon^i)^T(\epsilon^i)$. From the results presented in the next sub-section, it similarly follows that $\Omega^{ii}, \Sigma^{ii}$ are also non-vanishing if and only if $\epsilon$ is.

In fact, following [25], we can argue that there are actually no points in the spacetime where $K \equiv K^{ii}$ and hence $\epsilon^i$ vanishes. We will use two facts which will be discussed later: firstly that $K$ is either timelike or null and secondly that $K$ is Killing. Suppose then, that $K$ and $\epsilon^i$ vanish at a point $p$. Consider a time-like geodesic through $p$ with tangent vector $U$. Since $K$ is Killing, $U \cdot K$ is constant along the geodesic and since it vanishes at $p$ it must in fact vanish along the whole of the geodesic. However, $U$ is timelike while $K$ is non-spacelike and hence $K$ and thus $\epsilon^i$ must vanish along the whole of the geodesic. As this argument applies to any timelike geodesic passing through $p$ we conclude that $\epsilon^i$ vanishes to the future and past of the point $p$. Assuming analyticity, we conclude that $\epsilon^i$ vanishes everywhere (assuming that the spacetime is connected) which contradicts the assumption that we have a Killing spinor.



### 2.1 Algebraic relations

These differential forms defined above are not all independent. They satisfy certain algebraic relations which are a consequence of the underlying Clifford algebra. One way of obtaining these is by repeated use of Fierz identities. Another approach will be mentioned later. Let us illustrate this by considering the case $i = j$ and dropping the $ij$ indices, which covers the most general case when there is only one Killing spinor.

We first relate $\Omega^2$ and $\Sigma^2$ to $K^2$. We use here the convention that for any p-form $\alpha$ we have:

$$\alpha^2 \equiv \frac{1}{p!}\alpha_{\mu_1\mu_2\cdots\mu_p}\alpha^{\mu_1\mu_2\cdots\mu_p}\,.$$

By performing Fierz rearrangements on $K^2$, $\Omega^2$ and $\Sigma^2$ in turn we find three linearly dependent equations. Solving them we find:

$$\begin{aligned} \Sigma^2 &= -6K^2 \\ \Omega^2 &= -5K^2\,. \end{aligned} \tag{2.9}$$

We also find the following relations:

$$\Omega_{\mu_1}{}^{\sigma_1}\Omega_{\sigma_1}{}^{\nu_1} = -K_{\mu_1}K^{\nu_1} + \delta_{\mu_1}{}^{\nu_1}K^2 \tag{2.10}$$

$$\frac{1}{4!}\Sigma_{\mu_1}{}^{\sigma_1\sigma_2\sigma_3\sigma_4}\Sigma_{\sigma_1\sigma_2\sigma_3\sigma_4}{}^{\nu_1} = 14K_{\mu_1}K^{\nu_1} - 4\delta_{\mu_1}{}^{\nu_1}K^2 \tag{2.11}$$

$$i_K\Omega = 0 \tag{2.12}$$

$$i_K\Sigma = \frac{1}{2}\Omega\wedge\Omega \tag{2.13}$$

$$K^\sigma(*\Sigma)_{\sigma\nu_1\nu_2\nu_3\nu_4\nu_5} = \Omega_{\nu_1}{}^\sigma\Sigma_{\sigma\nu_2\nu_3\nu_4\nu_5} \tag{2.14}$$

$$K^2\Omega\wedge\Sigma = \frac{1}{2}K\wedge\Omega\wedge\Omega\wedge\Omega\,. \tag{2.15}$$

These are by no means exhaustive.

### 2.2 Differential relations

The covariant derivatives of the differential forms can be calculated by using the fact that a Killing spinor satisfies:

$$\overline{\nabla_\mu \epsilon^i} = \frac{1}{288}\bar{\epsilon}^i[\Gamma_\mu{}^{\nu_1\nu_2\nu_3\nu_4} + 8\delta_\mu^{\nu_1}\Gamma^{\nu_2\nu_3\nu_4}]F_{\nu_1\nu_2\nu_3\nu_4}\,. \tag{2.16}$$

We find

$$\begin{aligned}
\nabla_\mu K^{ij}_\nu &= \frac{1}{6}\Omega^{ij\sigma_1\sigma_2}F_{\sigma_1\sigma_2\mu\nu} + \frac{1}{6!}\Sigma^{ij\sigma_1\sigma_2\sigma_3\sigma_4\sigma_5} * F_{\sigma_1\sigma_2\sigma_3\sigma_4\sigma_5\mu\nu} \\
\nabla_\mu \Omega^{ij}_{\nu_1\nu_2} &= \frac{1}{3\cdot 4!}g_{\mu[\nu_1}\Sigma^{ij}{}_{\nu_2]}{}^{\sigma_1\sigma_2\sigma_3\sigma_4}F_{\sigma_1\sigma_2\sigma_3\sigma_4} + \frac{1}{3\cdot 3!}\Sigma^{ij}{}_{\nu_1\nu_2}{}^{\sigma_1\sigma_2\sigma_3}F_{\mu\sigma_1\sigma_2\sigma_3} - \\
&\quad - \frac{1}{3\cdot 3!}\Sigma^{ij}{}_{\mu[\nu_1}{}^{\sigma_1\sigma_2\sigma_3}F_{\nu_2]\sigma_1\sigma_2\sigma_3} + \frac{1}{3}K^{ij\ \sigma}F_{\sigma\mu\nu_1\nu_2} \\
\nabla_\mu \Sigma^{ij}_{\nu_1\nu_2\nu_3\nu_4\nu_5} &= \frac{1}{6}K^{ij\ \sigma} * F_{\sigma\mu\nu_1\nu_2\nu_3\nu_4\nu_5} - \frac{10}{3}F_{\mu[\nu_1\nu_2\nu_3}\Omega^{ij}_{\nu_4\nu_5]} - \frac{5}{6}F_{[\nu_1\nu_2\nu_3\nu_4}\Omega^{ij}_{\nu_5]\mu} - \\
&\quad - \frac{10}{3}g_{\mu[\nu_1}\Omega^{ij}{}_{\nu_2}{}^{\sigma}F_{|\sigma|\nu_3\nu_4\nu_5]} + \frac{5}{6}F_{\mu[\nu_1|\sigma_1\sigma_2|}(*\Sigma^{ij})^{\sigma_1\sigma_2}{}_{\nu_2\nu_3\nu_4\nu_5]} + \\
&\quad + \frac{5}{6}F_{[\nu_1\nu_2|\sigma_1\sigma_2|}(*\Sigma^{ij})^{\sigma_1\sigma_2}{}_{\nu_3\nu_4\nu_5]\mu} - \frac{5}{9}g_{\mu[\nu_1}F_{\nu_2|\sigma_1\sigma_2\sigma_3|}(*\Sigma^{ij})^{\sigma_1\sigma_2\sigma_3}{}_{\nu_3\nu_4\nu_5]}\,.
\end{aligned} \tag{2.17}$$

The exterior derivatives of the forms are thus given by

$$dK^{ij} = \frac{2}{3}i_{\Omega^{ij}}F + \frac{1}{3}i_{\Sigma^{ij}} * F \tag{2.18}$$

$$d\Omega^{ij} = i_{K^{ij}}F \tag{2.19}$$

$$d\Sigma^{ij} = i_{K^{ij}} * F - \Omega^{ij}\wedge F\,, \tag{2.20}$$

where e.g. $(i_\Omega F)_{\mu\nu} = (1/2!)\Omega^{\rho_1\rho_2}F_{\rho_1\rho_2\mu\nu}$.

From the first equation in (2.17) we can immediately deduce the important result that each of the $K^{ij}$ are Killing vectors. Moreover, using the Bianchi identity, it is simple to show that

$$\mathcal{L}_{K^{ij}}F = 0 \tag{2.21}$$

for any $K^{ij}$. Thus any geometry $(g, F)$ admitting Killing spinors posesses symmetries generated by $K^{ij}$.

Notice, as somewhat of an aside, that using (2.18) we also have

$$\mathcal{L}_{K^{ij}} * F = i_{K^{ij}}\left(d * F + \frac{1}{2}F\wedge F\right)\,. \tag{2.22}$$

Now the fact that $K^{ij}$ is Killing and the condition (2.21) implies that both the left and right hand side must vanish separately. This means that the presence of a Killing spinor implies that some components of the equation of motion for the four-form are automatically satisfied.[4] Notice also that this calculation provides a check on the sign appearing in the Chern-Simons term in the $D = 11$ supergravity lagrangian, given the form of the Killing spinor equation and the conventions for the Clifford algebra.

[4] Note that we are ignoring the $X_8$ correction here. To fully consistently incorporate it one needs to also consider higher order correction terms to the Killing spinor equation. However, we note here that if $i_{K^{ij}}X_8 = 0$ the above equation is consistent.

We next note that (2.19) is strikingly similar to the notion of generalised calibration for static membranes introduced in [14] following [13]. Indeed consider the special case that $i = j$ and when $K = K^{ii}$ is a static Killing vector. Then taking into account (2.12) we see that (2.19) is exactly the same equation satisfied by a generalised calibration $\Omega$ for a membrane that was introduced in [14]. Recall that the significance of generalised calibrations is that they calibrate supersymmetric brane world-volumes in the presence of non-zero four-form flux. What we have shown here is that supersymmetric $D = 11$ geometries automatically give rise to generalised calibrations $\Omega$.

That one gets the same result, in this special case, either from $D = 11$ supergravity or from the world-volume theory using kappa-symmetry as in [14], is perhaps not that surprising since it is well known that the kappa-symmetry of the super-membrane implies the equations of motion of $D = 11$ supergravity [26, 27]. What is particularly inteteresting, though, is that the $D = 11$ supergravity result indicates that the notion of generalised calibrations might be extended to more general settings than that studied in [14]. Firstly, since (2.19) is valid when $K$ is not only static but more generally stationary, it suggests that the analysis of [14] can be straightforwardly extended to the stationary case, as assumed in that paper. Secondly, (2.19) is also valid when when $K$ is null and it should be very interesting to elucidate the physical interpretation of this from the world-volme point of view. Finally, when there is more than one Killing spinor, $K^{ij}$ with $i \neq j$ can also be spacelike. This latter case is at least partially related to the issue discussed at the end of [14, section II] concerning the fact that static supersymmetric branes can have some flat directions.

The notion of generalised calibrations for fivebranes is more complicated due to the fact that the fivebrane world-volume has a self-dual three-form, which is responsible for the fact that membranes can end on fivebranes. An initial investigation was undertaken in [15] for the case of static configurations, where it was argued that the generalised calibration for the five-brane is a pair consisting of a spatial five-form and two-form. For static $K^{ii}$ these correspond to the spatial part of $\Sigma$ and $\Omega$. The possibility of the five-form not being closed was considered in [15] and argued to be related to Wess-Zumino terms in the fivebrane worldvolume theory. That $\Omega$ might also not be closed was not considered in [15], but here we see that this is the general situation and it is not difficult to see that this is again related to Wess-Zumino terms in the fivebrane worldvolume theory. Moreover, our analysis reveals the correct differential expressions when $K^{ii}$ is stationary and also when it is null, the latter case again being particularly intriguing. As for membranes, there also seems to be an interesting generalisation for spacelike $K^{ij}$ with $i \neq j$.

Returning to (2.17), it is also useful to note that we can extract

$$
\begin{aligned}
(*d*\Omega^{ij})_\nu &= -\frac{1}{3\cdot 4!}(\Sigma^{ij})_\nu{}^{\sigma_1\sigma_2\sigma_3\sigma_4}F_{\sigma_1\sigma_2\sigma_3\sigma_4} \\
(*d*\Sigma^{ij})_{\nu_1\nu_2\nu_3\nu_4} &= \frac{8}{3}(\Omega^{ij})^\sigma{}_{[\nu_1}F_{\nu_2\nu_3\nu_4]\sigma} + \frac{2}{9}(*\Sigma^{ij})^{\sigma_1\sigma_2\sigma_3}{}_{[\nu_1\nu_2\nu_3}F_{\nu_4]\sigma_1\sigma_2\sigma_3}\,. \qquad (2.23)
\end{aligned}
$$

Finally, using the algebraic results of the last subsection it is simple to conclude that the

Lie-derivative of $\Omega$ and $\Sigma$ with respect to $K$ vanish:

$$\begin{aligned}\mathcal{L}_K\Omega &= 0\\ \mathcal{L}_K\Sigma &= 0\,.\end{aligned} \tag{2.24}$$

The corresponding equations for $X, Y$ and $Z$ are given by

$$\begin{aligned}\nabla_\mu X^{ij} &= -\frac{1}{3\cdot 3!}(Y^{ij})^{\rho_1\rho_2\rho_3}F_{\rho_1\rho_2\rho_3\mu_1}\\ \nabla_\mu Y^{ij}_{\nu_1\nu_2\nu_3} &= -\frac{1}{3}X^{ij}F_{\mu\nu_1\nu_2\nu_3} - \frac{1}{6\cdot 3!}Y^{ij\,\rho_1\rho_2\rho_3} * F_{\rho_1\rho_2\rho_3\mu\nu_1\nu_2\nu_3} -\\ &\quad -\frac{1}{4}Z^{ij\,\rho_1\rho_2}{}_{\mu[\nu_1}F_{\nu_2\nu_3]\rho_1\rho_2} - \frac{1}{2}Z^{ij\,\rho_1\rho_2}{}_{[\nu_1\nu_2}F_{\nu_3]\mu\rho_1\rho_2} - \frac{1}{6}g_{\mu[\nu_1}Z^{ij\,\rho_1\rho_2\rho_3}_{\nu_2}F_{\nu_3]\rho_1\rho_2\rho_3}\\ \nabla_\mu Z^{ij}_{\nu_1\nu_2\nu_3\nu_4} &= \frac{2}{3}Y^{ij}{}_{\mu[\nu_1}{}^{\rho}F_{\nu_2\nu_3\nu_4]\rho} - 2Y^{ij}{}_{[\nu_1\nu_2}{}^{\rho}F_{\nu_3\nu_4]\mu\rho} -\\ &\quad - g_{\mu[\nu_1}Y^{ij}{}_{\nu_2}{}^{\rho_1\rho_2}F_{\nu_3\nu_4]\rho_1\rho_2} - \frac{1}{9} * Z^{ij}{}_{\mu[\nu_1\nu_2\nu_3}{}^{\rho_1\rho_2\rho_3}F_{\nu_4]\rho_1\rho_2\rho_3} +\\ &\quad +\frac{1}{18} * Z^{ij}{}_{\nu_1\nu_2\nu_3\nu_4}{}^{\rho_1\rho_2\rho_3}F_{\mu\rho_1\rho_2\rho_3} + \frac{1}{36}g_{\mu[\nu_1} * Z^{ij}{}_{\nu_2\nu_3\nu_4]}{}^{\rho_1\rho_2\rho_3\rho_4}F_{\rho_1\rho_2\rho_3\rho_4}\end{aligned} \tag{2.25}$$

and

$$\begin{aligned}(dX^{ij})_{\mu_1} &= -\frac{1}{3\cdot 3!}(Y^{ij})^{\rho_1\rho_2\rho_3}F_{\rho_1\rho_2\rho_3\mu_1}\\ (dY^{ij})_{\mu_1\mu_2\mu_3\mu_4} &= -\frac{1}{9}(Y^{ij})^{\rho_1\rho_2\rho_3}(*F)_{\rho_1\rho_2\rho_3\mu_1\mu_2\mu_3\mu_4} + (Z^{ij})_{[\mu_1\mu_2}{}^{\rho_1\rho_2}F_{\mu_3\mu_4]\rho_1\rho_2} -\\ &\quad -\frac{4}{3}X^{ij}F_{\mu_1\mu_2\mu_3\mu_4}\\ (dZ^{ij})_{\mu_1\mu_2\mu_3\mu_4\mu_5} &= -\frac{20}{3}(Y^{ij})_{[\mu_1\mu_2}{}^{\rho_1}F_{\mu_3\mu_4\mu_5]\rho_1} - \frac{5}{18}(*Z^{ij})_{[\mu_1\mu_2\mu_3\mu_4}{}^{\rho_1\rho_2\rho_3}F_{\mu_5]\rho_1\rho_2\rho_3}\end{aligned} \tag{2.26}$$

and also

$$\begin{aligned}(*d*Y^{ij})_{\nu_1\nu_2} &= 0\\ (*d*Z^{ij})_{\nu_1\nu_2\nu_3} &= -\frac{1}{2}(Y^{ij})^{\sigma_1\sigma_2}{}_{[\nu_1}F_{\nu_2\nu_3]\sigma_1\sigma_2} - \frac{1}{36}(*Z^{ij})_{\nu_1\nu_2\nu_3}{}^{\sigma_1\sigma_2\sigma_3\sigma_4}F_{\sigma_1\sigma_2\sigma_3\sigma_4}\,.\end{aligned} \tag{2.27}$$

### 2.3 Integrability

The integrability of the Killing spinor equation allows us to relate geometries admitting Killing spinors to those that in addition solve the equations of motion. As shown in the appendix, integrability of the Killing spinor equation implies that

$$\begin{aligned}0 &= \left[R_{\mu\nu} - \frac{1}{12}\left(F_{\mu\sigma_1\sigma_2\sigma_3}F_\nu{}^{\sigma_1\sigma_2\sigma_3} - \frac{1}{12}g_{\mu\nu}F^2\right)\right]\Gamma^\nu\epsilon^i -\\ &\quad -\frac{1}{6\cdot 3!} * \left(d*F + \frac{1}{2}F\wedge F\right)_{\sigma_1\sigma_2\sigma_3}\left(\Gamma_\mu{}^{\sigma_1\sigma_2\sigma_3} - 6\delta^{\sigma_1}_\mu\Gamma^{\sigma_2\sigma_3}\right)\epsilon^i -\\ &\quad -\frac{1}{6!}dF_{\sigma_1\sigma_2\sigma_3\sigma_4\sigma_5}\left(\Gamma_\mu{}^{\sigma_1\sigma_2\sigma_3\sigma_4\sigma_5} - 10\delta^{\sigma_1}_\mu\Gamma^{\sigma_2\sigma_3\sigma_4\sigma_5}\right)\epsilon^i\end{aligned} \tag{2.28}$$

for each Killing spinor $\epsilon^i$.

Assume that we have a geometry $(g, F)$ that admits Killing spinors and that also solves the Bianchi identity and the equations of motion for the four-form $F$. We then deduce from (2.28) that

$$0 = E_{\mu\nu}\Gamma^\nu\epsilon^i = 0\,, \qquad (2.29)$$

where $E_{\mu\nu} = 0$ is equivalent to the Einstein equations. We now follow the analysis of [9]: hitting (2.28) this with $\bar{\epsilon}^i$ we conclude that

$$E_{\mu\nu}K^\nu = 0\,. \qquad (2.30)$$

On the other hand if we hit it with $E_{\mu\sigma}\Gamma^\sigma$ we conclude that

$$E_{\mu\nu}E_\mu{}^\nu = 0 \qquad \text{no sum on} \quad \mu\,. \qquad (2.31)$$

As we shall discuss, the Killing vector $K \equiv K^{ii}$ is either a timelike or null. We first assume that it is timelike. Introducing an orthonormal frame with $K = e^0$, we deduce from (2.30) that $E_{\mu 0} = 0$. The indices in (2.31) then run over spatial indices only and we conclude that $E_{\mu\nu} = 0$ since there are no non-trivial null vectors in a euclidean space. Alternatively if $K$ is null we can set up a $D = 11$ frame

$$ds^2 = 2e^+e^- + e^a e^a \qquad (2.32)$$

for $a = 1, \ldots 9$, with $K = e^+$. Now (2.30) implies $E_{-\mu} = 0$ while (2.31) implies $E_{+a} = E_{ab} = 0$. Hence, one just needs to impose $E_{++} = 0$ to obtain a full supersymmetric solution.

These results have some obvious practical benefits in finding explicit solutions.

## 3. Classifying solutions using $G$-structures

In the last section we derived a number of necessary conditions, both algebraic and differential, for a geometry to possess Killing spinors. A useful organisational principle is that of a $G$-structure.

Let us begin by recalling the definition of $G$-structure of a $n$-dimensional manifold $M$. The frame bundle is a principal $\mathrm{Gl}(n)$ bundle and a $G$-structure is simply a principal $G$-sub-bundle. Often, the $G$-structure can be equivalently specified by the existence of nowhere vanishing $G$-invariant tensors, and it is in this geometric guise that $G$-structures will be important here. For example, a metric of euclidean signature is equivalent to an $O(n)$ structure and if supplemented with an orientation is equivalent to an $\mathrm{SO}(n)$ structure. An almost complex structure $J$ is equivalent to a $\mathrm{Gl}(n/2, C)$ structure, and if supplemented with an hermitean metric is equivalent to a $\mathrm{U}(n/2)$ structure, and so on. In $D = 11$ supergravity the manifolds are equipped with a lorentzian metric, and a spin structure, so the frame bundle can always be reduced to $\mathrm{Spin}(10, 1)$ and hence there is always a $\mathrm{Spin}(10, 1)$ structure.

Let us explain the main ideas in classifying $G$-structures using $G \subset \mathrm{Spin}(10, 1)$ as an example (see e.g. [28]–[30] for further discussion). Consider a $G \subset \mathrm{Spin}(10, 1)$ structure specified by $G$-invariant tensors and/or spinors, that we collectively define by $\eta$. The essential idea is simple: as $G$ defines a metric, one can take the covariant derivative of $\eta$ with respect to the Levi-Civita connection and then decompose the result into irreducible $G$-modules. In more detail, one first uses the fact that there is no obstruction to finding a connection preserving the structure. If we choose one, $\nabla'$, then one notes that $\nabla\eta = (\nabla - \nabla')\eta$. Now $(\nabla - \nabla')$ is a tensor with values in $T^* \otimes spin(10,1)$ but acting on the $G$-invariant $\eta$ we see that the piece of $\nabla\eta$ that is independent of $\nabla'$ is given by an element of $T^* \otimes g^\perp$ where $g \oplus g^\perp = spin(10,1)$. This element is known as the intrinsic torsion and can be decomposed into irreducible $G$-modules: $\nabla\eta \leftrightarrow T^* \otimes g^\perp = \oplus_i \mathcal{W}_i$. In one extreme, all of these modules $\mathcal{W}_i$ are present and one has the most general type of $G$-structure. In the other extreme, all of the modules vanish and the tensors are covariantly constant giving rise to manifolds with special holonomy $G$.

We can now use this language to interpret the algebraic and differential conditions that we obtained for Killing spinor bi-linears in the last section. In particular, it will provide us with a framework for classifying the local form of all supersymmetric solutions of $D = 11$ supergravity.

Start with the most general supersymmetric geometry preserving (at least) one Killing spinor. That is, start with a $D = 11$ geometry with a $\mathrm{Spin}(10,1)$ structure and assume that we have a globally defined spinor, which is equivalently specified by the tensors $K, \Omega, \Sigma$ constructed from the bi-linears in the Killing spinors that we introduced in the last section. We showed in section 2 that the Killing spinor is non-vanishing. At a point, the isotropy group of the spinor is known to be either $\mathrm{SU}(5)$ or $(\mathrm{Spin}(7) \ltimes \mathbb{R}^8) \times \mathbb{R}$ [1] depending on whether $K$ is time-like or null, respectively. A spacelike $K$ is not possible. The forms $K, \Omega, \Sigma$ are invariant under $\mathrm{SU}(5)$ or $(\mathrm{Spin}(7) \ltimes \mathbb{R}^8) \times \mathbb{R}$ in each case, and one can interpret the complete algebraic identities that $K, \Omega, \Sigma$ satisfy, of which we obtained a subset in the last section, as simply encoding this information.

In the "null"-case $K$ is null everywhere and in the "time-like" case it is time-like at least at a point and hence at least in a neighbourhood of such a point. In the null case the Killing spinor, or equivalently $K, \Omega, \Sigma$, define a $D = 11$ $(\mathrm{Spin}(7) \ltimes \mathbb{R}^8) \times \mathbb{R}$ structure. A description of this unusual structure can be found in [1] (see also [2]).

For the time-like case, in situations in which $K$ is time-like everywhere, the Killing spinor, or equivalently $K, \Omega, \Sigma$, define a $D = 11$ $\mathrm{SU}(5)$ structure. It can happen, however, that there are supersymmetric geometries in which $K$ is time-like in some regions but becomes null at some points in the manifold, such as at a horizon. In the generic case, therefore, we observe that if we restrict to regions in which $K$ is time-like then $K, \Omega, \Sigma$ still defines an $\mathrm{SU}(5)$ structure. It is in this sense that we will say that the timelike case has an $\mathrm{SU}(5)$ structure. This will be sufficient for our purposes, but we comment that since $(K, \Omega, \Sigma)$ are globally defined there may be a another mathematical language to describe this situation.

The analysis in the next section for the time-like case, will be carried out in a neighbourhood in which the Killing vector is timelike. Since the neighbourhood is topologically trivial, the frame bundle can always be trivialised. To avoid possible confusion, we emphasise that the Killing spinor defines a privileged $D = 11$ $\mathrm{SU}(5)$ structure which satisfies certain differential conditions and these are non-trivial even in the topologically trivial neighbourhood. For example the privileged SU(5) structure is equivalent to a certain metric on this neighbourhood.

So any geometry admitting a $D = 11$ Killing spinor will either have a SU(5) structure or a $(\mathrm{Spin}(7) \ltimes \mathbb{R}^8) \times \mathbb{R}$ structure that is built from the Killing spinor. The next steps in classifying the local form of the most general supersymmetric bosonic geometries are to determine, in each case, the intrinsic torsion of the $G$-structure i.e., the type of $G$-structure that arises according to the classification of $G$-structures, and then to see how much of the form of the geometry is specified by the structure. This can be achieved by analysing the differential conditions imposed on the tensors $K, \Omega, \Sigma$ that we obtained from the Killing spinor equation earlier.

We will carry out this analysis in detail for the case of a single timelike Killing spinor in the next section. We will see that the differential conditions restrict the type of SU(5) structure. In addition we will determine the general local form of the geometry and show that it is almost completely determined by the SU(5) structure: we will see that there is a component of the four-form that is not fixed by supersymmetry alone. We will also prove a converse result, namely that given such a SU(5) structure, with the appropriately specified four-form, then the geometry does indeed admit at least one Killing spinor.

As noted in the last section, for the timelike case the integrability conditions for the Killing spinor imply that in order to have a supersymmetric solution to the equations of motion, one just needs to impose the Bianchi identity and the equations of motion for the four-form. These conditions impose further independent constraints and in particular constrain the component of the four-form not specified by supersymmetry.

A similar analysis for the null case, which we will leave for future work, would then complete a classification of the local form of the most general types of bosonic $D = 11$ supergravity solutions. A finer classification would be to find the additional restrictions placed on these geometries in order that they preserve more than one supersymmetry. The additional Killing spinors will either be time-like or null and hence there will be several SU(5) and/or $(\mathrm{Spin}(7) \ltimes \mathbb{R}^8) \times \mathbb{R}$ structures. Equivalently, the structure group can be reduced further to the maximal common subgroup which is the isotropy group of all of the spinors. So a first step to obtain this refined classification would be to classify all of the different isotropy groups of $1, \ldots, 32$ spinors. In principle, one way of tackling this problem would be to derive algebraic conditions on the tensors $K^{ij}, \Omega^{ij}, \Sigma^{ij}$ and $X^{ij}, Y^{ij}, Z^{ij}$ using various Fierz identities. However, the calculations we carried out for the tensors with $i = j$ in the last section were already very involved and this would be a very clumsy approach. It should be more efficient to generalise the work of [1]. In addition, we note that the groups appearing in [2, tables 1 and 2] will certainly be relevant.

The second step in obtaining the refined classification would be to determine the intrinsic torsion of the $G$-structure defined by the Killing spinors and to see how much of the form of the geometry is specified by the structure. Again this information is contained in the differential conditions imposed on the tensors $K^{ij}, \Omega^{ij}, \Sigma^{ij}$ and $X^{ij}, Y^{ij}, Z^{ij}$ that we obtained from the Killing spinor equation. The results of the next section on the case of a single time-like spinor are encouraging that this entire programme could be carried out.

## 4. The stationary case and SU(5) structure

In this section we will determine the most general form of geometries admitting at least one timelike Killing spinor. The spinor can be used to construct a one-form $K$, a two-form $\Omega$ and a five-form $\Sigma$. Working in a neighbourhood where $K$ is time-like, these forms together specify a privileged SU(5) structure in $D = 11$. An important restriction on this $D = 11$ structure is that the dual time-like vector field to $K$ is Killing. We can thus introduce a time coordinate along the orbits of the Killing vector, so that we have $K = -\Delta^2(dt + \omega)$, with $\Delta$ and $\omega$ independent of $t$. The metric then takes the form:

$$ds_{11}^2 = -\Delta^2(dt + \omega)^2 + \Delta^{-1} g_{mn} dx^m dx^n \tag{4.1}$$

and $K^2 = -\Delta^2$. The metric $\Delta^{-1} g_{mn}$ is a metric on the ten dimensional euclidean spatial base manifold, which we will denote by $B$, defined via the orthogonal projection of the eleven dimensional metric with respect to the Killing vector. We will work in these coordinates and then obtain the full geometry by analytic continuation at the end.

From (2.12) and (2.24) we immediately deduce that $\Omega$ is a two-form on the base manifold. If we raise an index using the metric $g$ we obtain an almost complex structure on $B$. The metric $g$ is then hermitean with respect to this almost complex structure and $\Omega$ is the Kähler form.

Using (2.13) it follows that the five-form $\Sigma$ can be written as

$$\Sigma = \frac{1}{2}\Delta^{-1} e^0 \wedge \Omega \wedge \Omega + \Delta^{-3/2}\chi\,, \tag{4.2}$$

where, again using (2.24), $\chi$ is a five-form on $B$ and we have defined

$$e^0 = \Delta(dt + \omega) \tag{4.3}$$

which can be used to build an orthonormal frame in $D = 11$. If we now define

$$\theta = \chi - i * \chi\,, \tag{4.4}$$

where, in this section, $*$ is the Hodge star with respect to the metric $g$, we conclude from (2.14) that $\theta$ is a $(5, 0)$ form on $B$. This means that the ten-dimensional base manifold $B$ admits an SU(5) structure specified by $\Omega, \theta$, or equivalently by $g, \Omega, \chi$. In most of the subsequent analysis, the focus will be on the D=10 SU(5) structure on $B$.

Note that the factors of $\Delta$ were inserted in the definition of $\chi$ in (4.2) to ensure that the SU(5) structure satisfies the compatibility condition

$$\chi \wedge *\chi = -2^4 \frac{\Omega^5}{5!}\,. \tag{4.5}$$

Note also that

$$\chi^2 = 16\,, \tag{4.6}$$

where the indices here are contracted using the metric $g$, which can be deduced from (2.9).

The existence of an SU(5) structure allows us to decompose the complexified space of forms on $B$ into irreducible representations of SU(5), and this will be very useful in the following. We first decompose the space of forms into $(p,q)$-forms. Pure forms of type $(p,0)$ form irreducible representations of SU(5). For mixed forms we need to remove traces taken with $\Omega$ to form irreducible representations so these split further into:

$$\begin{aligned}
\Lambda^{(1,1)} &\cong \Lambda_0^{(1,1)} \oplus \mathbb{R} \\
\Lambda^{(2,1)} &\cong \Lambda_0^{(2,1)} \oplus \Lambda^{(1,0)} \\
\Lambda^{(2,2)} &\cong \Lambda_0^{(2,2)} \oplus \Lambda_0^{(1,1)} \oplus \mathbb{R} \\
\Lambda^{(3,1)} &\cong \Lambda_0^{(3,1)} \oplus \Lambda^{(2,0)} \\
\Lambda^{(3,2)} &\cong \Lambda_0^{(3,2)} \oplus \Lambda_0^{(2,1)} \oplus \Lambda^{(1,0)} \\
\Lambda^{(4,1)} &\cong \Lambda_0^{(4,1)} \oplus \Lambda^{(3,0)} \,,
\end{aligned} \tag{4.7}$$

where the subscript 0 denotes a traceless form. The rest are determined by complex conjugation and by noting that * maps a $(p,q)$-form to $(5-q,5-p)$-form.

It will be helpful at this point to briefly review the classification of SU(5) structures on ten-dimensional riemannian manifolds (further comments are made in appendix C). We noted in the last section that $G$-structures are classified by the intrinsic torsion, which is an element of $T^* \otimes g^\perp$. Here $g^\perp$ is defined by $su(5) \oplus g^\perp \cong so(10)$. Noting that the adjoint of $so(10)$ decomposes under $su(5)$ via $\mathbf{45} \to \mathbf{1} + \mathbf{10} + \bar{\mathbf{10}} + \mathbf{24}$ and that $\mathbf{24}$ is the adjoint of $su(5)$, we conclude that the intrinsic torsion is given by the SU(5) modules:

$$(\mathbf{5}+\bar{\mathbf{5}}) \times (\mathbf{1}+\mathbf{10}+\bar{\mathbf{10}}) \to (\mathbf{10}+\bar{\mathbf{10}}) + (\mathbf{40}+\bar{\mathbf{40}}) + (\mathbf{45}+\bar{\mathbf{45}}) + (\mathbf{5}+\bar{\mathbf{5}}) + (\mathbf{5}'+\bar{\mathbf{5}}') \,. \tag{4.8}$$

In other words, the intrinsic torsion is given by five SU(5) modules: $T^* \otimes g^\perp \simeq \mathcal{W}_1 \oplus \mathcal{W}_2 \oplus \mathcal{W}_3 \oplus \mathcal{W}_4 \oplus \mathcal{W}_5$, where conventionally[5] the $\mathcal{W}_i$ are given in the order noted in (4.8). The component of the intrinsic torsion in the module $\mathcal{W}_i$ will be denoted by $W_i$.

It will be very important in the following to use the fact that the $W_i$, and hence the intrinsic torsion, are determined by $d\Omega$ and $d\chi$. One sees that this is possible by consideration of the $su(5)$ irreps appearing in $d\Omega$ and $d\chi$. Consider first the three-form $d\Omega$ corresponding to the $\mathbf{120}$ of SO(10). Since $\Omega$ is a $(1,1)$ form, $d\Omega$ will have a $(3,0)+(0,3)$ piece and also a $(2,1)+(1,2)$ piece. Removing the trace from the latter pieces, one obtains the decomposition $\mathbf{120} \to \mathbf{45} + \bar{\mathbf{45}} + \mathbf{10} + \bar{\mathbf{10}} + \mathbf{5} + \bar{\mathbf{5}}$ under SU(5) $\subset$ SO(10). Similarly, since $\chi$ is the real part of a $(5,0)$ form, the six-form $d\chi$ will have a $(5,1)+(1,5)$ and a $(4,2)+(2,4)$ part. These give rise to the representations $\mathbf{5}+\bar{\mathbf{5}}+\mathbf{10}+\bar{\mathbf{10}}+\mathbf{40}+\bar{\mathbf{40}}$. We thus see that $d\Omega$ and $d\chi$ contain all the irreps appearing in the $W_i$. In more detail we can define the following irreducible components of $d\Omega$ and $d\chi$:

$$\chi \wedge d\Omega = \Omega \wedge d\chi = W_1 \wedge \frac{\Omega^3}{3!}$$

[5]The modules $\mathcal{W}_1 \oplus \mathcal{W}_2 \oplus \mathcal{W}_3 \oplus \mathcal{W}_4$ are those arising in the classification of U(5) structures [31].



JHEP04(2003)039

$$
\begin{aligned}
(d\chi)^{4,2} + (d\chi)^{2,4} &= W_2 \wedge \Omega + \frac{1}{3} W_1 \wedge \frac{\Omega^2}{2!} \\
(d\Omega)^{2,1} + (d\Omega)^{1,2} &= W_3 + \frac{1}{4} W_4 \wedge \Omega \\
W_4 &= \Omega \lrcorner d\Omega \\
W_5 &= \chi \lrcorner d\chi
\end{aligned}
\tag{4.9}
$$

with $W_1 = *(\Omega \wedge d\chi)$. Here we have introduced the notation $\omega \lrcorner \nu$ which contracts a $p$-form $\omega$ into a $n+p$-form $\nu$ via:

$$
(\omega \lrcorner \nu)_{i_1 \dots i_n} = \frac{1}{p!} \omega^{j_1 \dots j_p} \nu_{j_1 \dots j_p i_1 \dots i_n} \,. \tag{4.10}
$$

A more precise connection between the intrinsic torsion and the $W_i$ is presented in appendix C. Note that the $\mathbf{10} + \bar{\mathbf{10}}$ part of $d\chi$ is related to the $\mathbf{10} + \bar{\mathbf{10}}$ of $d\Omega$ through the condition $\Omega \wedge \chi = 0$. For orientation, note that the almost complex structure is integrable iff $W_1 = W_2 = 0$ so that manifolds with an SU(5)-structure of type $\mathcal{W}_3 \oplus \mathcal{W}_4 \oplus \mathcal{W}_5$ are hermitean manifolds. Also if all $W_i$ vanish so that the intrinsic torsion vanishes then the manifold is Ricci flat and has holonomy $G \subseteq \mathrm{SU}(5)$. We will see that the SU(5) structure arising on the base manifold $B$ is only weakly restricted in general.

We are now ready to relate the components of $F$ to the SU(5)-structure $(g, \Omega, \chi)$. We will see that almost all of $F$ is determined by the structure. First we write

$$
F = \Delta^{-1} e^0 \wedge G + H \,, \tag{4.11}
$$

where $G$ is a three-form and $H$ is a four-form defined on $B$. The eleven dimensional Hodge dual of $F$ is thus given by:

$$
*_{11} F = \Delta^{-3} * G + \Delta^{-1} e^0 \wedge *H \,.
$$

We find from (2.19) that

$$
G = d\Omega \tag{4.12}
$$

and from (2.20) that

$$
d(\Delta^{-3/2}\chi) + \frac{1}{2} d\omega \wedge \Omega \wedge \Omega = *H - \Omega \wedge H \,. \tag{4.13}
$$

So $G$ is clearly determined by the structure. We now attempt to solve (4.13) for $H$. Introducing the map

$$
\begin{aligned}
\Theta : \Lambda^4(B) &\to \Lambda^4(B) \\
\alpha &\mapsto \alpha - *(\Omega \wedge \alpha)
\end{aligned}
\tag{4.14}
$$

we can rewrite (4.13) as:

$$
* \,\Theta(H) = d(\Delta^{-3/2}\chi) + \frac{1}{2} d\omega \wedge \Omega \wedge \Omega \,. \tag{4.15}
$$

| Irreps | E-Values |
|---|---|
| $\mathbf{1}$ | $-2$ |
| $\mathbf{5}, \bar{\mathbf{5}}$ | $0$ |
| $\mathbf{10}, \bar{\mathbf{10}}$ | $-1$ |
| $\mathbf{24}$ | $3$ |
| $\mathbf{40}, \bar{\mathbf{40}}$ | $2$ |
| $\mathbf{75}$ | $0$ |

**Table 1:** Eigenvalues of the irreducible representations of $\Lambda^6$ under the map $\Theta$. Since the map is real the conjugates of the complex representations have the same eigenvalues.

Now $H$ is a four-form and can be split into irreducible SU(5) representations. The **210** of SO(10) decomposes under SU(5) as,

$$\mathbf{210} \to \mathbf{1} + \mathbf{5} + \bar{\mathbf{5}} + \mathbf{10} + \bar{\mathbf{10}} + \mathbf{24} + \mathbf{40} + \bar{\mathbf{40}} + \mathbf{75}\,.$$

We can thus write

$$H = H_1 + H_{5+\bar{5}} + H_{10+\bar{10}} + H_{24} + H_{40+\bar{40}} + H_{75}\,. \tag{4.16}$$

Each four-form $H_i$ can then be written in terms of certain $(p, q)$ forms defining SU(5) irreps. Using the identities listed in (D.1) a calculation shows that each irreducible representation is an eigenvector of $\Theta$ with eigenvalues given in table 1. Note that three of the representations have zero eigenvalue. This has important consequences, as we shall see. We next split the right hand side of (4.15) into irreducible SU(5) components. For $d\chi$ this was noted above. For the second term we write $d\omega = d\omega^{(0)}\Omega + d\omega_0^{(1,1)} + d\omega^{(2,0)} + d\omega^{(0,2)}$, corresponding to the decomposition $\mathbf{45} \to \mathbf{1} + \mathbf{24} + \mathbf{10} + \bar{\mathbf{10}}$.

So let us see what we can conclude from the above. First consider the **75** part. This is projected out on the left hand side of (4.15) but is also not present on the right hand side. So we have no contradiction here. Next consider the $\mathbf{5}+\bar{\mathbf{5}}$ part. Again this is projected out by $\Theta$ but is generically present on the left hand side. So we conclude that the $(5, 1)$ piece of $d(\Delta^{-3/2}\chi)$ vanishes. Equivalently, we conclude that the $(5, 1)$ piece of $d\chi$ corresponding to $W_5$ is exact:

$$W_5 = -12\, d \log \Delta\,. \tag{4.17}$$

For the remaining representations the eigenvalues are non-zero and (4.15) allows us to determine the corresponding $H_i$ in terms of the structure. We find,

$$\begin{aligned}
H_1 &= -\frac{3}{4}(d\omega)^{(0)}\Omega^2 \\
H_{10+\bar{10}} &= -\left[\frac{1}{3} * (\Omega \wedge d(\Delta^{-3/2}\chi)) + (d\omega)^{(2,0)} + (d\omega)^{(0,2)}\right] \wedge \Omega \\
H_{24} &= -\frac{1}{3}(d\omega)_0^{(1,1)} \wedge \Omega \\
H_{40+\bar{40}} &= \frac{1}{2} * d(\Delta^{-3/2}\chi) - \frac{1}{6} * (\Omega \wedge d(\Delta^{-3/2}\chi)) \wedge \Omega\,,
\end{aligned} \tag{4.18}$$

where $\Omega^n$ denotes the wedge product of $n$ factors of $\Omega$.

At this point both $H_{75}$ and $H_{5+\bar{5}}$ are undetermined. However, we can now use the equation for $dK$ in (2.18) to relate the $\mathbf{5}+\bar{\mathbf{5}}$ part of $d\Omega$ and $H$ to $\Delta$ giving:

$$\frac{\Delta^{3/2}}{6\ 5!} * H_{mn_1\dots n_5}\chi^{n_1\dots n_5} = \partial_m log\Delta - \frac{1}{6}\Omega^{r_1r_2}(d\Omega)_{mr_1r_2}\,. \tag{4.19}$$

Looking at equations (2.23) we find no further constraints.

So let us summarize what we have learned. Any geometry admitting a timelike Killing spinor can be written in the form

$$ds^2_{11} = -\Delta^2(dt+\omega)^2 + \Delta^{-1}g_{mn}dx^m dx^n\,, \tag{4.20}$$

where the base space with metric $g$ admits an SU(5) structure $(g,\Omega,\chi)$ whose only restriction is that $W_5$ is exact and related to the warp factor $\Delta$ via

$$W_5 = -12 d\log\Delta\,. \tag{4.21}$$

The four-form field strength can be written as:

$$\begin{aligned} F \;=\; & (dt+\omega)\wedge d\Omega - \left[\frac{3}{4}(d\omega)^{(0)}\Omega + (d\omega)^{(2,0)} + (d\omega)^{(0,2)} + \frac{1}{3}(d\omega)^{(1,1)}_0\right]\wedge\Omega + \\ & + \frac{1}{2} * d(\Delta^{-3/2}\chi) - \frac{1}{2} * \left[\Omega\wedge d\left(\Delta^{-3/2}\chi\right)\right]\wedge\Omega - \\ & - \frac{1}{16} * \left([W_5+4W_4]\wedge\Delta^{-3/2}\chi\right) + F_{75}\,, \end{aligned} \tag{4.22}$$

where $F_{75}$ $(= H_{75})$ is an arbitrary 4-form on $B$ in the $\mathbf{75}$ of SU(5) (i.e. $F_{75}\in\Lambda_0^{(2,2)}$), $W_4$ and $W_5$ are defined in (4.9) and $d\omega = d\omega^{(0)}\Omega + d\omega_0^{(1,1)} + d\omega^{(2,0)} + d\omega^{(0,2)}$.

We started this section with the SU(5) structure in $D=11$ specified by $K,\Omega,\Sigma$. However, our derivation of (4.20)–(4.22) mostly involved the SU(5) structure in D=10 which is a component of the $D=11$ SU(5) structure. The reason for this is that the $D=11$ structure is constrained by the fact that $K$ is Killing, $\mathcal{L}_K\Omega = \mathcal{L}_K\Sigma = 0$ and we worked with the obvious adapted co-ordinates. Now $d\omega$ is an arbitrary closed[6] two-form on $B$ as far as the D=10 SU(5) structure is concerned. On the other hand, one can show that $dK$ specifies a part of the intrinsic torsion of the $D=11$ structure, and since $dK = 2d(log\Delta)\wedge K - \Delta^2 d\omega$ we conclude that $d\omega$ is in fact determined by the $D=11$ structure. This is in contrast to $F_{75}$ which is determined by neither the $D=10$ nor the $D=11$ structure. See appendix E for further comments about the type of SU(5) structure in $D=11$.

We have thus derived necessary conditions on the local form of the most general bosonic geometry admitting a timelike Killing spinor. The form includes a completely undetermined component of the field strength and one might wonder if further conditions might be imposed by considering the covariant derivatives of $\Omega$ and $\Sigma$ and not just the exterior derivatives. To see that they are not we will now prove a converse result: that the above form of the metric and four-form field strength does indeed admit a Killing spinor. In

[6]Since $\omega$ need only be locally defined.

particular $F_{75}$ drops out of the Clifford connection appearing in the Killing spinor equation. It is for this reason that this component of the field-strength is not determined by the Killing spinor equation alone.

To see this we first note that the geometry should preserve a single Killing spinor $\epsilon$ giving rise to the SU(5) structure in $D = 11$. Such a spinor can be specified by demanding that it be left inert by a number of projection operators. First introduce the obvious orthonormal frame

$$
\begin{aligned}
e^0 &= \Delta(dt+\omega) \\
e^i &= \Delta^{-1/2}\bar{e}^i\,,
\end{aligned}
\tag{4.23}
$$

where $\bar{e}^i$ is an orthonormal frame for the base manifold $B$. The $D = 11$ gamma matrices give rise to D=10 gamma-matrices $\Gamma^i$ with $\Gamma^0 = \Gamma_{1\dots10}$ proportional to the chirality operator. It is convenient to introduce the chiral complex spinor

$$
\eta = \frac{(1+i\Gamma_0)}{\sqrt{2}}\epsilon \tag{4.24}
$$

in terms of which

$$
\begin{aligned}
\Omega_{ij} &= -i\eta^\dagger\Gamma_{ij}\eta \\
\theta_{i_1\dots i_5} &= i\eta^T\Gamma_{i_1\dots i_5}\eta\,.
\end{aligned}
\tag{4.25}
$$

Both the real an imaginary parts of $\eta$ give equivalent SU(5) structures, but only the real part will be a Killing spinor, as we shall see. Now the almost complex structure on the base manifold is not integrable in general, and hence we cannot always introduce complex coordinates. Nevertheless we can consistently introduce holomorphic and anti-holomorphic tangent space indices which simplifies the calculation. In terms of these we conclude that $\eta$ satisfies the projections

$$
\Gamma^a\eta = 0\,, \tag{4.26}
$$

where $a = 1,\dots,5$ is a holomorphic index and also

$$
\Gamma_{a_1\dots a_5}\eta = -i\theta_{a_1\dots a_5}\eta^*\,. \tag{4.27}
$$

We now consider the Killing spinor equation acting on $\epsilon = (\eta+\eta^*)/\sqrt{2}$. Plugging in the expression for the four-form and dealing with each SU(5) irrep separately, we find after a lengthy computation and using (D.2), that

$$
\begin{aligned}
\Big[\nabla_m &+ \frac{1}{160}(\Omega W_5 + 5\Omega W_4)_m\Omega_{k_1k_2}\Gamma^{k_1k_2} - \frac{1}{16}(W_4)_k{\Gamma_m}^k + \\
&+\frac{1}{8}{\Omega_m}^r(W_3)_{rk_1k_2}\Gamma^{k_1k_2} - \frac{1}{394}{\chi_{mk_1k_2}}^{n_1n_2}(W_1)_{n_1n_2}\Gamma^{k_1k_2} + \\
&+\frac{1}{192}{\Omega_m}^r(W_2)_{r\ell_1\ell_2\ell_3}{\chi^{\ell_1\ell_2\ell_3}}_{k_1k_2}\Gamma^{k_1k_2}\Big](\eta_0+\eta_0^*) = 0\,,
\end{aligned}
\tag{4.28}
$$

where we have rescaled the spinor $\eta \equiv \Delta^{1/2}\eta_0$ and used the notation $\Omega V_m \equiv \Omega_m{}^r V_r$. Now both $\eta_0$ and $\eta_0^*$ are solutions to this equation since the connection is simply the sum of the Levi-Civita connection on $B$ with the intrinsic contorsion of the $D = 10$ SU(5) structure, as we show in appendix C. However, one should not conclude that there are two Killing spinors: the point is that (4.28) only arises when the Killing spinor equation is acting on the sum $(\eta+\eta^*)/\sqrt{2}$ and not on $\eta, \eta^*$ separately. Thus we conclude that the geometry in general preserves one Killing spinor $\Delta^{1/2}(\eta_0 + \eta_0^*)$ corresponding to just 1/32 supersymmetry.

Note that while the covariant derivative appearing in the original Killing spinor equation of $D = 11$ supergravity (2.3) takes values in the Clifford algebra, the covariant derivative appearing in (4.28) takes values in the spin sub-algebra. In other words, the four-form field strength is necessarily constrained in such a way that it transforms the Clifford connection into a spin connection when acting on the preserved supersymmetries.

We have now shown that the form (4.20), (4.21), (4.22) is both necessary and sufficient for a geometry to admit a single time-like Killing spinor. However not all such spacetimes are solutions of eleven dimensional supergravity. To obtain solutions of the theory one just has to impose the gauge equations of motion and the Bianchi identity for $F$ since the Einstein equations will then be automatically be satisfied as we showed in section 2.3. The Bianchi identity for $F$ can be written

$$d\omega \wedge d\Omega + dH = 0 \tag{4.29}$$

while the equation of motion for the four-form gives rise to two equations

$$\begin{aligned} d(\Delta^{-3} * d\Omega) + d\omega \wedge *H + \frac{1}{2}H \wedge H &= -\beta X_8 \\ d\Omega \wedge H - d * H &= 0\,. \end{aligned} \tag{4.30}$$

Here we have added in the correction term to the field equation and have assumed that $i_K X_8 = 0$. Note that the third equation is actually implied by the first (see the discussion following (2.22)). To check this in detail one can take the exterior derivative of (4.15) to find

$$d(*\Theta(H)) - d\omega \wedge d\Omega \wedge \Omega = 0 \tag{4.31}$$

and then substitute (4.29). Note that one can further substitute the expression for $H$ given by (4.18) and (4.19) into (4.29), (4.30) but as the result is not too illuminating we shall not present it here. It is worth emphasising that the component of the four-form not determined by the Killing spinor equation, $F_{75}$, is constrained and related to the SU(5) structure by the Bianchi identity and the equations of motion

At this stage we can present some vanishing theorems when the ten-dimensional base manifold $B$ is compact. Consider first the case when $H = 0$ and hence $F = (dt + \omega) \wedge d\Omega$. We then have

$$\int_B \Delta^{-3} d\Omega \wedge *d\Omega = -\int_B \Omega \wedge d(*\Delta^{-3} d\Omega) = 0\,, \tag{4.32}$$

where we have integrated by parts and then used the equation of motion (4.30) (ignoring the $X_8$ correction). Since the left hand side of the equation is positive semi-definite we conclude that $d\Omega = 0$ which in turn implies that the four-form $F = 0$.

Let us now obtain two results for non-vanishing $H$. First observe that using (4.15) and (4.18) we deduce

$$d(\Delta^{-3/2}\chi) = *\Theta(H) - \frac{1}{2}d\omega\wedge\Omega^2 = *\Theta(H') \tag{4.33}$$

where $H'$ is defined to be the pieces of $H$ that are independent of $d\omega$. Next consider

$$\int_B *\Theta(H')\wedge\Theta(H') = -\int_B \Delta^{-3/2}\chi\wedge d\Theta(H')\,, \tag{4.34}$$

where we have integrated by parts. We next note that

$$d\Theta(H') = d(-H'_{10+\bar{10}} + 2H_{40+\bar{40}})\,. \tag{4.35}$$

On the other hand we know from the Bianchi identity (4.29) that

$$d\omega\wedge d\Omega + dH = 0\,. \tag{4.36}$$

If we now restrict to $d\omega = d\omega^{(2,0)} + d\omega^{(0,2)}$ then this equation becomes

$$d\left(H_{5+\bar{5}} - H'_{10+\bar{10}} + H_{40+\bar{40}} + H_{75}\right) = 0\,. \tag{4.37}$$

Comparing with (4.35) suggests we should further restrict to $H_{5+\bar{5}} = H_{75} = 0$.

We now obtain two theorems. If $d\omega = d\omega^{(2,0)}+d\omega^{(0,2)}$, $H_{5+\bar{5}} = H_{75} = 0$ and $H_{40+\bar{40}} = 0$ then (4.37), (4.35) and (4.33) implies that $H_{10+\bar{10}} = 0$ also and hence $H$=0. Similarly if $d\omega = d\omega^{(2,0)} + d\omega^{(0,2)}$, $H_{5+\bar{5}} = H_{75} = 0$ and $H_{10+\bar{10}} = 0$ then $H_{40+\bar{40}} = 0$ also and hence $H$=0. In both cases the previous result assuming $H = 0$ and arbitrary $d\omega$ then implies that $F = 0$.

## 5. Examples of solutions with SU(5) structures

In order to gain some further insight into the formalism, we will now display SU(5) structures for some known solutions. As a bonus, in carrying out this exercise we will be able to spot some new solutions.

### 5.1 M5 branes

Let us first look at the simple $M5$-brane solution. The metric and field strength can be written as:

$$\begin{aligned} ds^2_{11} &= H^{-1/3}(-dt^2 + dx^idx^i) + H^{2/3}dy^idy^i \\ *_{11}F &= -dt\wedge dx^1\wedge dx^2\wedge dx^3\wedge dx^4\wedge dx^5\wedge dH^{-1} \end{aligned} \tag{5.1}$$

with $i = 1,\ldots,5$ and $H = H(y)$ a harmonic function. This solution is well known to preserve 1/2 of the supersymmetry: the Killing spinors satisfy the single projection $\Gamma_{012345}\epsilon = \epsilon$. There are certainly timelike spinors which satisfy this projection and so we should be able to display a SU(5) structure for it.

Comparing with (4.20) we identify $\Delta = H^{-1/6}$ and the base space metric is then given by,

$$ds^2 = H^{-1/2}dx^i dx^i + H^{1/2}dy^i dy^i \tag{5.2}$$

Define the complex $(1,0)$ frame,

$$\Theta^i = H^{-1/4}dx^i + iH^{1/4}dy^i\,. \tag{5.3}$$

The corresponding SU(5) structure is given by:

$$\begin{aligned}\Omega &= \frac{i}{2}\Theta^i \wedge \bar{\Theta}^i \\ \chi &= \mathrm{Re}(\Theta^1 \wedge \Theta^2 \wedge \Theta^3 \wedge \Theta^4 \wedge \Theta^5)\,.\end{aligned} \tag{5.4}$$

In terms of the coordinate basis these are given by

$$\begin{aligned}\Omega &= dx^i \wedge dy^i \\ \chi &= H^{-5/4}dx^{12345} - \frac{H^{-1/4}}{3!2!}\varepsilon_{i_1\ldots i_5}dx^{i_1i_2i_3} \wedge dy^{i_4i_5} + \frac{H^{3/4}}{4!}\varepsilon_{i_1\ldots i_5}dx^{i_1} \wedge dy^{i_2\ldots i_5}\,,\end{aligned} \tag{5.5}$$

where $\varepsilon_{i_1\ldots i_5}$ is just the $d = 5$ permutation symbol.

To see that this SU(5) structure is indeed related to a Killing spinor we first note, after a small calculation, that $d(\Delta^{-3/2}\chi)$ has no $(5,1) + (1,5)$ pieces and hence (4.21) is satisfied. This is the only restriction required on the structure, but we note that here we also have $d\Omega = 0$. We next need to show that the four-form can be recovered from (4.22). Interestingly, to achieve this it is necessary to include a non-vanishing $F_{75}$. Specifically we set

$$F_{75} = \frac{1}{16 \cdot 2!2!}\partial^i logH \varepsilon_{ij_1\ldots j_4}\Theta^{j_1} \wedge \Theta^{j_2} \wedge \bar{\Theta}^{j_3} \wedge \bar{\Theta}^{j_4} \tag{5.6}$$

and then (4.22) agrees with the expression in (5.1).

Since the fivebrane solution preserves 16 Killing spinors the solution has more than one SU(5) structure. Note also that some of the Killing spinors can be null so that the solution also belongs to the null class. It would be interesting to display the SU(5) structure for the solution corresponding to a fivebrane wrapping a SLAG five-cycle [32], as this solution preserves just 1/32 supersymmetry .

### 5.2 Flat and resolved $M2$ branes

Let us now recover some known solutions involving membranes. We take the ten-dimensional base space $B$ to be of the form:

$$ds^2 = \Delta^3(dx_1^2 + dx_2^2) + ds^2(M_8)\,, \tag{5.7}$$

where $ds^2(M_8)$ is a Ricci flat metric with holonomy contained in SU(4). One can then define the following SU(5) structure

$$\begin{aligned}\Omega &= \Delta^3 dx^1 \wedge dx^2 + \omega_{(8)} \\ \chi &= \Delta^{3/2}(dx^1 \wedge \hat{\chi}_1 + dx^2\hat{\chi}_2)\,,\end{aligned} \tag{5.8}$$

where we have introduced the Kähler form $\omega_{(8)}$ and holomorphic $(4,0)$ form $\hat{\theta} = \hat{\chi}_1 - i\hat{\chi}_2$ of $M_8$. The base space has in fact an SU(4) $\subset$ SU(5) structure. Note that the normalizations of $\Omega$ and $\chi$ are not arbitrary but are chosen to ensure that $\Omega$ is a Kähler form for the base space and that they satisfy the compatibility condition (4.5). Also we could have chosen an arbitrary function $f^2$ in the metric instead of $\Delta^3$ but demanding that (4.21) is satisfied implies that $f^2 = \Delta^3$.

For simplicity we will assume that $\Delta$ does not depend on $(x^1, x^2)$. This implies that the $\mathbf{5} + \bar{\mathbf{5}}$ piece of the spatial part of the four-form field strength vanishes. Let us first restrict to static solutions and set the rotation parameter to zero. The expression for the four-form field strength (4.22) becomes

$$F = dt \wedge dx^1 \wedge dx^2 \wedge d\Delta^3 + F_{75}\,, \tag{5.9}$$

where $F_{75}$ is any four form on the base space in the $\mathbf{75}$ of SU(5). Imposing the Bianchi identity and gauge equations of motion and using $*F_{75} = F_{75} \wedge \Omega$ one finds:

$$\begin{aligned} dF_{75} &= 0 \\ d *_8 d\Delta^{-3} - \frac{1}{2} F_{75} \wedge F_{75} &= -\beta X_8\,, \end{aligned} \tag{5.10}$$

where $*_8$ is the Hodge star with respect to the 8 dimensional metric.

In the simple case that $F_{75} = 0$ and flat transverse space, we recover the well known 1/2 supersymmetric $M2$ brane solution

$$\begin{aligned} ds^2 &= H^{-2/3}(-dt^2 + dx_1^2 + dx_2^2) + H^{1/3} ds^2(\mathbb{E}^8) \\ F &= dt \wedge dx^1 \wedge dx^2 \wedge d(H^{-1})\,, \end{aligned} \tag{5.11}$$

where $H \equiv \Delta^{-3}$ is harmonic.

Another possibility is to take $F_{75}$ to be a four form on $M_8$. Under SU(4) $\subset$ SU(5) we have the following decomposition: $\mathbf{75} \to \mathbf{15} + \mathbf{20} + \bar{\mathbf{20}} + \mathbf{20'}$. The first three representations occur when one of the indices of $F_{75}$ is in the $(x^1, x^2)$ directions so for $F_{75}$ to be a four form on $M_8$ it must belong to the $\mathbf{20'}$ of SU(4) i.e. it must be a self-dual $(2,2)$ form. Since it must be closed it follows that $F_{75} = L_{(2,2)}$ with $L_{(2,2)}$ a harmonic self dual four form. This modifies the equation for $H$ and we get

$$\Box H = -\frac{1}{2} |L_{(2,2)}|^2 + \beta X_8\,. \tag{5.12}$$

Thus we recover the resolved 1/8 supersymmetric $M2$ brane solutions of [18]–[22]. As for the fivebrane solution, these solutions also belong to the null class.

A simple rotating generalisation of these solutions is to choose $d\omega$ to be the sum of a $(2,0) + (0,2)$ form. Specifically, given a closed two-form $\alpha \in \Lambda^{2,0}(M_8)$ we set $d\omega = \alpha + \bar{\alpha}$ and get the supersymmetric solution

$$\begin{aligned} ds^2 &= H^{-2/3}[-(dt + \omega)^2 + dx_1^2 + dx_2^2] + H^{1/3} ds^2(M_8) \\ F &= (dt + \omega) \wedge dx^1 \wedge dx^2 \wedge d(H^{-1}) + F_{75} \end{aligned} \tag{5.13}$$

with (5.10) unchanged.

### 5.3 Rotating Calabi-Yau and the Gödel solution

Recently, an interesting generalisation of the Gödel solution was found in five-dimensions [9]. Uplifted to $D = 11$ it was shown to preserve 5/8 of the supersymmetry. We now show that this fits into a broader class of new solutions.

We look for rotating solutions with no warp factor for which the base space $M_{10}$ is a complex manifold with holonomy $G \subseteq \mathrm{SU}(5)$. Then, the only non-zero components of the field strength can arise from the rotation and from $F_{75}$. Similarly to the last sub-section, we set $d\omega = \alpha + \bar{\alpha}$ with $\alpha \in \Lambda^{2,0}(M_{10})$. We then find the supersymmetric solution

$$\begin{aligned} ds^2_{11} &= -(dt+\omega)^2 + ds^2(M_{10}) \\ F &= -d\omega \wedge \Omega + F_{75} \end{aligned} \tag{5.14}$$

provided that $dF_{75} = 0$ and $F_{75} \wedge F_{75} = 2\beta X_8$.

As a particular example of this class of solutions we take the base space to be flat $\mathbb{E}^{10}$ and set $F_{75} = 0$. Introduce complex coordinates $z^a$ for $\mathbb{E}^{10}$ and the canonical SU(5) structure,

$$\begin{aligned} \Omega &= \frac{i}{2} dz^a \wedge d\bar{z}^a \\ \chi &= \mathrm{Re}(dz^1 \wedge dz^2 \wedge dz^3 \wedge dz^4 \wedge dz^5)\,. \end{aligned} \tag{5.15}$$

One can then choose $\alpha = dz^1 \wedge dz^2$ and this gives the Gödel solution of [9] which preserves 5/8 supersymmetry. Further Gödel-type solutions are obtained by choosing different $\alpha \in \Lambda^{2,0}(\mathbb{E}^{10})$, and it would be interesting to see which of them preserve exotic fractions of supersymmetry.

## 6. Conclusion

We have shown that the most general supersymmetric configurations of $D = 11$ supergravity, preserving at least one Killing spinor, have either a privileged SU(5) or a $(\mathrm{Spin}(7) \ltimes \mathbb{R}^8) \times \mathbb{R}$ structure constructed from the Killing spinor, depending on whether the vector constructed from a bi-linear of the Killing spinor is time-like or null, respectively. For the time-like case, we carried out a detailed local analysis using the Killing spinor equation: we found that the SU(5) structure in $D = 11$ is restricted by the fact that the time-like vector is always Killing and that the SU(5) structure of the D=10 base space orthogonal to the orbits of the Killing vector is only weakly constrained. We deduced the necessary and sufficient conditions on the form of the geometry admitting Killing spinors and showed that most of the form is determined by the $D = 11$ SU(5) structure. In particular there was a component of the four-form which dropped out of the Killing spinor equation and hence is undetermined. We also analysed what extra constraints are imposed in order to ensure that the geometries preserving Killing spinors also solve the equations of motion. These constraints relate the component of the four-form undetermined by the Killing spinor equation to the SU(5) structure. To complete the classification of the most general supersymmetric solutions we need to carry out a similar analysis for the null case.

The formalism was used to prove some no-go theorems when the D=10 base space is compact and also to construct some new solutions. It would be interesting to study the new solutions further. For example it would be interesting to determine the fraction of supersymmetry preserved by the new Gödel-type solutions presented at the end of section 5.3.

We have also proposed a finer classification for configurations that preserve more than one supersymmetry. Such configurations will have various numbers of different SU(5) and/or $(\mathrm{Spin}(7)\ltimes\mathbb{R}^8)\times\mathbb{R}$ structures that can be constructed from the Killing spinors, or equivalently, a privileged $G$-structure with $G\subset\mathrm{SU}(5)$ or $G\subset(\mathrm{Spin}(7)\ltimes\mathbb{R}^8)\times\mathbb{R}$. The first step then, is to classify these $G$-structures which are defined to be the isotropy groups of the Killing spinors. They can also be specified by algebraic conditions on the tensors that can be constructed from bi-linears in the spinors. The second step in the classification is to then use the Killing spinor equation to place constraints on the $G$-structure, as well as to solve for various parts of the metric and four-form field strength in terms of the structure. It would be quite an achievement to carry out this programme in full.

## Acknowledgments

We would like to thank Chris Hull for collaboration in the early stages of this work and, with Daniel Waldram, for very helpful discussions. We would also like to thank José Figueroa-O'Farrill for useful comments on an earlier draft.

## A. Conventions

We use the signature $(-,+,\ldots,+)$. $D=11$ co-ordinate indices will be denoted $\mu,\nu,\ldots$ while tangent space indices will be denoted by $\alpha,\beta,\ldots$.

The $D=11$ spinors we will use are Majorana. The gamma matrices satisfy

$$\{\Gamma_\alpha,\Gamma_\beta\}=2\eta_{\alpha\beta} \tag{A.1}$$

and can be taken to be real in the Majorana representation. They satisfy, in our conventions, $\Gamma_{012345678910}=1$ and as a consequence of this the following duality relation holds:

$$\Gamma_{\alpha_1\alpha_2\ldots\alpha_p}=(-1)^{\frac{(p+1)(p-2)}{2}}\frac{1}{(11-p)!}\varepsilon_{\alpha_1\alpha_2\ldots\alpha_p}{}^{\alpha_{p+1}\alpha_{p+2}\ldots\alpha_{11}}\Gamma_{\alpha_{p+1}\alpha_{p+2}\ldots\alpha_{11}}\,, \tag{A.2}$$

where we have defined

$$\varepsilon_{012345678910}=+1\,. \tag{A.3}$$

It follows that $\Delta=\{1,\Gamma_{\alpha_1},\Gamma_{\alpha_1\alpha_2},\Gamma_{\alpha_1\alpha_2\alpha_3},\Gamma_{\alpha_1\alpha_2\alpha_3\alpha_4},\Gamma_{\alpha_1\alpha_2\alpha_3\alpha_4\alpha_5}\}$ is a basis of the Clifford algebra $C\ell(10,1)\cong R(32)$, where $R(32)$ is the algebra of $32\times 32$ matrices. We will use repeatedly the following formula for anti-symmetrising products of gamma matrices:

$$\begin{aligned}\Gamma_{\alpha_1\alpha_2\ldots\alpha_n}\Gamma^{\beta_1\beta_2\ldots\beta_m}&=\\ &=\sum_{k=0}^{k=\min(n,m)}\frac{m!n!}{(m-k)!(n-k)!k!}\Gamma_{[\alpha_1\alpha_2\ldots\alpha_{n-k}}{}^{[\beta_{k+1}\ldots\beta_m}\delta^{\beta_1}_{\alpha_n}\delta^{\beta_2}_{\alpha_{n-1}}\cdots\delta^{\beta_k]}_{\alpha_{n-k+1}]}\,.\end{aligned} \tag{A.4}$$

For any $M, N \in R(32)$ we can perform a Fierz rearrangement using:

$$\begin{aligned} M_a{}^b N_c{}^c = \frac{1}{32}\Big\{ & (NM)_a{}^d \delta_c{}^b + (N\Gamma^{\alpha_1} M)_a{}^d (\Gamma_{\alpha_1})_c{}^b - \\ & - \frac{1}{2!}(N\Gamma^{\alpha_1\alpha_2} M)_a{}^d (\Gamma_{\alpha_1\alpha_2})_c{}^b - \frac{1}{3!}(N\Gamma^{\alpha_1\alpha_2\alpha_3} M)_a{}^d (\Gamma_{\alpha_1\alpha_2\alpha_3})_c{}^b + \\ & + \frac{1}{4!}(N\Gamma^{\alpha_1\alpha_2\alpha_3\alpha_4} M)_a{}^d (\Gamma_{\alpha_1\alpha_2\alpha_3\alpha_4})_c{}^b + \\ & + \frac{1}{5!}(N\Gamma^{\alpha_1\alpha_2\alpha_3\alpha_4\alpha_5} M)_a{}^d (\Gamma_{\alpha_1\alpha_2\alpha_3\alpha_4\alpha_5})_c{}^b \Big\}, \end{aligned}$$

where $a, b, c, d = 1, \ldots, 32$.

Given a Majorana spinor $\epsilon$ its conjugate is given by $\bar{\epsilon} = \epsilon^T C$, where $C$ is the charge conjugation matrix in $D = 11$ and satisfies $C^T = -C$. In the Majorana representation we can choose $C = \Gamma_0$. An important property of gamma matrices in $D = 11$ is that the matrix $C\Gamma_{\alpha_1\alpha_2\ldots\alpha_p}$ is symmetric for $p = 1, 2, 5$ and antisymmetric for $p = 0, 3, 4$ (the cases $p > 5$ are related by duality to the above). For an antisymmetrized product $\Gamma_{(n)}$ of $n$ gamma matrices and any spinor $\epsilon$ we have :

$$\overline{\Gamma_{(n)}\epsilon} = (-1)^{\frac{n(n+1)}{2}} \bar{\epsilon}\Gamma_{(n)} \,. \tag{A.5}$$

The Hodge star of a $p$-form $\omega$ is defined by

$$*\omega_{\mu_1\ldots\mu_{11-p}} = \frac{\sqrt{-g}}{p!}\epsilon_{\mu_1\ldots\mu_{11-p}}{}^{\nu_1\ldots\nu_p}\omega_{\nu_1\ldots\nu_p} \tag{A.6}$$

and the square of a $p$-form via

$$\omega^2 = \frac{1}{p!}\omega_{\mu_1\ldots\mu_p}\omega^{\mu_1\ldots\mu_p} \,. \tag{A.7}$$

## B. Integrability conditions from the Killing spinor equation

Taking the second covariant derivative of the Killing spinor equation (2.3) and anti-symmetrising we obtain:

$$\begin{aligned} \nabla_{[\rho}\nabla_{\mu]}\epsilon = & -\frac{1}{288}(\Gamma_{[\mu}{}^{\nu_1\nu_2\nu_3\nu_4} - 8\delta^{\nu_1}_{[\mu}\Gamma^{\nu_2\nu_3\nu_4})\nabla_{\rho]}F_{\nu_1\nu_2\nu_3\nu_4}\epsilon + \\ & + \frac{1}{288^2}(\Gamma_{[\mu}{}^{\nu_1\nu_2\nu_3\nu_4} - 8\delta^{\nu_1}_{[\mu}\Gamma^{\nu_2\nu_3\nu_4})(\Gamma_{\rho]}{}^{\sigma_1\sigma_2\sigma_3\sigma_4} - 8\delta^{\sigma_1}_{[\rho}\Gamma^{\sigma_2\sigma_3\sigma_4})F_{\nu_1\nu_2\nu_3\nu_4}F_{\sigma_1\sigma_2\sigma_3\sigma_4}\epsilon \,. \end{aligned}$$

The terms on the right hand side can be simplified using the identity:

$$\begin{aligned} & (\Gamma_{[\mu}{}^{\nu_1\nu_2\nu_3\nu_4} - 8\delta^{\nu_1}_{[\mu}\Gamma^{\nu_2\nu_3\nu_4})(\Gamma_{\rho]}{}^{\sigma_1\sigma_2\sigma_3\sigma_4} - 8\delta^{\sigma_1}_{\rho]}\Gamma^{\sigma_2\sigma_3\sigma_4}) = \\ & = \Gamma_{\mu\rho}{}^{\nu_1\nu_2\nu_3\nu_4\sigma_1\sigma_2\sigma_3\sigma_4} + 16\delta^{\nu_1}_{[\mu}\Gamma_{\rho]}{}^{\nu_2\nu_3\nu_4\sigma_1\sigma_2\sigma_3\sigma_4} + 2\cdot 4!\delta^{\nu_1\sigma_1}_{\mu\rho}\Gamma^{\nu_2\nu_3\nu_4\sigma_2\sigma_3\sigma_4} + \\ & \quad + 4\cdot 4!\delta^{\nu_1}_{[\mu}g^{\nu_2\sigma_1}\Gamma_{\rho]}{}^{\nu_3\nu_4\sigma_2\sigma_3\sigma_4} - 3\cdot 4!g^{\nu_1\sigma_1}g^{\nu_2\sigma_2}\Gamma_{\mu\rho}{}^{\nu_3\nu_4\sigma_3\sigma_4} - 4!4!\delta^{\nu_1}_{[\mu}g^{\nu_2\sigma_1}g^{\nu_3\sigma_2}\Gamma_{\nu]}{}^{\nu_4\sigma_3\sigma_4} + \\ & \quad + 16\cdot 4!\delta^{\nu_1\nu_2}_{\mu\rho}g^{\nu_3\sigma_1}\Gamma^{\nu_4\sigma_2\sigma_3\sigma_4} + 4!g^{\nu_1\sigma_1}g^{\nu_2\sigma_2}g^{\nu_3\sigma_3}g^{\nu_4\sigma_4}\Gamma_{\mu\rho} - 8\cdot 4!\delta^{\nu_1}_{[\mu}g^{\nu_2\sigma_1}g^{\nu_3\sigma_2}g^{\nu_4\sigma_3}\Gamma_{\rho]}{}^{\sigma_4} - \\ & \quad - 36\cdot 4!\delta^{\nu_1\sigma_1}_{\mu\rho}g^{\nu_2\sigma_2}g^{\nu_3\sigma_3}\Gamma^{\nu_4\sigma_4} \qquad (AS)\,, \end{aligned} \tag{B.1}$$

where "AS" refers to the fact that this equation is true when we anti-symmetrise over the indices $\sigma_1, \ldots, \sigma_4$ and $\nu_1, \ldots, \nu_4$. Also, the left hand side can be expressed in terms of the Riemann tensor via

$$\nabla_{[\rho}\nabla_{\mu]}\epsilon = \frac{1}{8}R_{\rho\mu\sigma_1\sigma_2}\Gamma^{\sigma_1\sigma_2}\epsilon\,. \tag{B.2}$$

The resulting integrability condition first appeared in [33].

We note that the integrability condition can be recast in the form

$$\left(\sum_{r=1}^{r=5}\frac{1}{r!}L^{(r)\sigma_1\ldots\sigma_r}_{\mu\nu}\Gamma_{\sigma_1\ldots\sigma_r}\right)\epsilon = 0\,, \tag{B.3}$$

where we have,

$$\begin{aligned}
L^{(1)\sigma_1}_{\mu\nu} &= -\frac{1}{144} * (F\wedge F)_{\mu\nu}{}^{\sigma_1}\\
L^{(2)\sigma_1\sigma_2}_{\mu\nu} &= -\frac{1}{4}R_{\mu\nu}{}^{\sigma_1\sigma_2} - \frac{1}{72}F^2\delta^{\sigma_1}_{[\mu}\delta^{\sigma_2}_{\nu]} - \frac{1}{216}\delta^{\sigma_1}_{[\mu}F_{\nu]}{}^{\lambda_1\lambda_2\lambda_3}F_{\lambda_1\lambda_2\lambda_3}{}^{\sigma_2} +\\
&\quad + \frac{1}{48}F_{[\mu}{}^{\sigma_1\lambda_1\lambda_2}F_{\nu]\lambda_1\lambda_2}{}^{\sigma_2}\\
L^{(3)\sigma_1\sigma_2\sigma_3}_{\mu\nu} &= \frac{1}{6}\nabla_{[\mu}F_{\nu]}{}^{\sigma_1\sigma_2\sigma_3} + \frac{1}{216} * F_{[\mu}{}^{\sigma_1\sigma_2\sigma_3\lambda_1\lambda_2\lambda_3}F_{\nu]\lambda_1\lambda_2\lambda_3}\\
L^{(4)\sigma_1\sigma_2\sigma_3\sigma_4}_{\mu\nu} &= \frac{1}{9}F_{\mu\nu}{}^{\sigma_1\lambda_1}F_{\lambda_1}{}^{\sigma_2\sigma_3\sigma_4} - \frac{1}{6}\delta^{\sigma_1}_{[\mu}F_{\nu]}{}^{\sigma_2\lambda_1\lambda_2}F_{\lambda_1\lambda_2}{}^{\sigma_3\sigma_4}\\
L^{(5)\sigma_1\sigma_2\sigma_3\sigma_4\sigma_5}_{\mu\nu} &= \frac{5}{12}\delta^{\sigma_1}_{[\mu}\nabla_{\nu]}F^{\sigma_2\sigma_3\sigma_4\sigma_5} - \frac{1}{1728}\epsilon_{\lambda_1\lambda_2\lambda_3\lambda_4\lambda_5\lambda_6}{}^{\sigma_1\sigma_2\sigma_3\sigma_4\sigma_5}F_{[\mu}{}^{\lambda_1\lambda_2\lambda_3}F_{\nu]}{}^{\lambda_4\lambda_5\lambda_6} -\\
&\quad -\frac{1}{864}\epsilon_{\lambda_1\lambda_2\lambda_3\lambda_4\lambda_5[\mu}{}^{\sigma_1\sigma_2\sigma_3\sigma_4\sigma_5}F_{\nu]}{}^{\lambda_1\lambda_2\kappa_1}F_{\kappa_1}{}^{\lambda_3\lambda_4\lambda_5} +\\
&\quad + \frac{1}{1152}\epsilon_{\mu\nu\lambda_1\lambda_2\lambda_3\lambda_4}{}^{\sigma_1\sigma_2\sigma_3\sigma_4\sigma_5}F^{\lambda_1\lambda_2\kappa_1\kappa_2}F_{\kappa_1\kappa_2}{}^{\lambda_3\lambda_4}\,.
\end{aligned} \tag{B.4}$$

If we now contract with $\Gamma^\mu$ and use the Bianchi identity $R_{\mu[\nu\rho\sigma]} = 0$ we find that:

$$\begin{aligned}
0 = &\left[R_{\rho\mu} - \frac{1}{12}\left(F_{\rho\sigma_1\sigma_2\sigma_3}F_{\mu}{}^{\sigma_1\sigma_2\sigma_3} - \frac{1}{12}g_{\rho\mu}F^2\right)\right]\Gamma^\mu\epsilon -\\
&- \frac{1}{6\cdot 3!} * (d*F + \frac{1}{2}F\wedge F)_{\sigma_1\sigma_2\sigma_3}\left(\Gamma_\rho{}^{\sigma_1\sigma_2\sigma_3} - 6\delta^{\sigma_1}_\rho\Gamma^{\sigma_2\sigma_3}\right)\epsilon -\\
&- \frac{1}{6!}dF_{\sigma_1\sigma_2\sigma_3\sigma_4\sigma_5}(\Gamma_\rho{}^{\sigma_1\sigma_2\sigma_3\sigma_4\sigma_5} - 10\delta^{\sigma_1}_\rho\Gamma^{\sigma_2\sigma_3\sigma_4\sigma_5})\epsilon\,.
\end{aligned} \tag{B.5}$$

## C. SU(5) structures in ten dimensions

Consider a $SU(5) \subset SO(10)$ structure specified by $g, \Omega, \chi$ or equivalently by a chiral spinor $\eta$. As mentioned earlier, there always exists a connection $\nabla'$ that preserves the structure, $\nabla'\eta = 0$. In fact it is not unique and there is a whole family of such connections. The intrinsic torsion of the SU(5) structure is the part of the torsion of an SU(5) preserving connection that does not depend on the specific choice of such a connection. It can thus be thought of as an equivalence class of torsion tensors. Let us explain this in more detail.

Any metric preserving connection can be written as $\Gamma^r{}_{mn} = C^r{}_{mn} + K^r{}_{mn}$, where $C^r{}_{mn}$ are the Christoffel symbols and $K^r{}_{mn}$ is called the contorsion tensor. The contorsion satisfies the symmetry property $K_{rmn} = -K_{nmr}$ and the torsion can be determined by the contorsion by $T^r{}_{mn} = 2K^r{}_{[mn]}$. One can also construct the contorsion tensor from the torsion as

$$K^r{}_{mn} = \frac{1}{2}(T^r{}_{mn} + T_m{}^r{}_n + T_n{}^r{}_m)\,. \tag{C.1}$$

Thus the torsion and contorsion are essentially equivalent.

The contorsion (and torsion) tensor lives in the space $T^* \otimes so(10) \simeq (T^* \otimes su(5)) \oplus (T^* \otimes su(5)^\perp)$, where $su(5)^\perp$ is the orthogonal complement of $su(5)$ in $so(10)$. The part of the contorsion tensor that lies in $T^* \otimes su(5)$ acts trivially on SU(5) singlets such as $\Omega$ and $\chi$. Thus any two connections that preserve the SU(5) structure will differ by an element of $T^* \otimes su(5)$ and so the intrinsic contorsion, which we denote $K^0$, is the part of the contorsion that lies in $T^* \otimes su(5)^\perp$.

The space $T^* \otimes su(5)^\perp$, where the intrinsic contorsion lies, decomposes as

$$(\mathbf{5}+\bar{\mathbf{5}}) \times (\mathbf{1}+\mathbf{10}+\bar{\mathbf{10}}) \to (\mathbf{10}+\bar{\mathbf{10}}) + (\mathbf{40}+\bar{\mathbf{40}}) + (\mathbf{45}+\bar{\mathbf{45}}) + (\mathbf{5}+\bar{\mathbf{5}}) + (\mathbf{5}'+\bar{\mathbf{5}}') \tag{C.2}$$

while the space $T^* \otimes su(5)$ decomposes as,

$$(\mathbf{5}+\bar{\mathbf{5}}) \times (\mathbf{24}) \to (\mathbf{5}+\bar{\mathbf{5}}) + (\mathbf{45}+\bar{\mathbf{45}}) + (\mathbf{70}+\bar{\mathbf{70}})\,. \tag{C.3}$$

We thus see that the contorsion tensor has, generically, three $\mathbf{5}+\bar{\mathbf{5}}$ and two $\mathbf{45}+\bar{\mathbf{45}}$ pieces. The most general contorsion tensor can thus be written

$$\begin{aligned} K_{rmn} = {} & T^{(1)}{}_m\Omega_{nr} + T^{(2)}{}_{[n}\Omega_{r]m} + T^{(3)}{}_{[n}g_{r]m} + \left(P_0^{2,1}{}_{rmn} + \Omega_m{}^k Q_0^{2,1}{}_{knr} + \text{c.c.}\right) + \\ & + \left(S_0^{3,1}{}_{mk_1k_2k_3}\chi^{k_1k_2k_3}{}_{nr} + \text{c.c.}\right)(R^{3,0}_{mnr} + \text{c.c.}) + K^{70+\bar{70}}_{rmn}\,, \end{aligned} \tag{C.4}$$

where c.c. stands for complex conjugate, $P, Q, R, S$ are forms of the type indicated. Demanding that the above connection preserves the SU(5) structure allows one to relate the various components of the contorsion tensor to the $W_i$ defined in (4.9).

To do this let $\nabla'$ be a covariant derivative with contorsion $K$ that preserves the SU(5) structure. To proceed write $\nabla'\Omega = \nabla'\chi = 0$ and then anti-symmetrise all of the indices to get

$$\begin{aligned} \frac{1}{6}d\Omega_{n_1n_2n_3} &= K^r{}_{[n_1n_2}\Omega_{|r|n_3]} \\ \frac{1}{30}d\chi_{n_1\dots n_6} &= K^r{}_{[n_1n_2}\chi_{|r|n_3n_4n_5n_6]}\,. \end{aligned} \tag{C.5}$$

It is now possible to explicitly relate the irreps of $K$ to $W_i$ by decomposing the left and right hand sides into SU(5) irreps. We find that $R$ and $S$ are uniquely determined by $W_1$ and $W_2$, respectively, and that

$$\begin{aligned} (W_3)_{n_1n_2n_3} &= -6\Omega_{[n_1}{}^r P_0^{2,1}{}_{n_2n_3]r} - 2Q_0^{2,1}{}_{n_1n_2n_3} + \text{c.c.} \\ (W_4)_m &= -4\Omega_m{}^r T_r^{(2)} - 4T_m^{(3)} \\ (W_5)_m &= 40\Omega_m{}^r T_r^{(1)} + 12\Omega_m{}^r T_r^{(2)} + 20T^{(3)}\,. \end{aligned} \tag{C.6}$$

The fact that the above components of the contorsion are not uniquely determined in terms of the $W_i$ reflects the freedom in defining an SU(5) preserving connection. Solving for $T^{(1)}, T^{(2)}$ in terms of $W_4, W_5, T^{(3)}$ and for $P_0^{2,1}$ in terms of $W_3, Q_0^{2,1}$ we conclude that the contorsion can be expressed as

$$\begin{aligned} K_{rmn} &= -\frac{1}{40}(\Omega W_5 + 3\Omega W_4)_m \Omega_{nr} + \frac{1}{4}(\Omega W_4)_{[n}\Omega_{r]m} + \frac{3}{2}\Omega_{[r}{}^k (W_3)_{mn]k} + \\ &+ \frac{1}{4\cdot 4!}\chi_{rmn}{}^{k_1k_2}(W_1)_{k_1k_2} - \frac{1}{2\cdot 4!}\Omega_m{}^k (W_2)_{k\ell_1\ell_2\ell_3}\chi^{\ell_1\ell_2\ell_3}{}_{nr} + \\ &+ \left(\frac{1}{5}\Omega T^{(3)}_m \Omega_{nr} + \Omega T^{(3)}{}_{[n}\Omega_{r]m} + T^{(3)}{}_{[n}g_{r]m}\right) + \\ &+ \left(3\Omega_{[r}{}^k Q_0^{2,1}{}_{mn]k} + \Omega_m{}^k Q_0^{2,1}{}_{nrk} + \text{c.c.}\right) + K^{70+\overline{70}}_{rmn}\,, \end{aligned} \tag{C.7}$$

where we have used the notation $\Omega V_m \equiv \Omega_m{}^r V_r$. The last three terms in the brackets act trivially on $\Omega$ and $\chi$ and so correspond to the terms appearing in the decomposition (C.3). Thus the intrinsic contorsion can be defined by (C.7) with the last three terms set to zero. We thus have

$$\begin{aligned} K^0_{rmn} &= -\frac{1}{40}(\Omega W_5 + 3\Omega W_4)_m \Omega_{nr} + \frac{1}{4}(\Omega W_4)_{[n}\,\Omega_{r]m} + \frac{3}{2}\Omega_{[r}{}^k (W_3)_{mn]k} + \\ &+ \frac{1}{4\cdot 4!}\chi_{rmn}{}^{k_1k_2}(W_1)_{k_1k_2} - \frac{1}{2\cdot 4!}\Omega_m{}^k (W_2)_{k\ell_1\ell_2\ell_3}\chi^{\ell_1\ell_2\ell_3}{}_{nr}\,. \end{aligned} \tag{C.8}$$

Equivalently we can calculate from this the intrinsic torsion thus showing that it is fully determined by the $W_i$, as claimed. For completeness we record the explicit form:

$$\begin{aligned} T^0_{rmn} &= \frac{1}{20}(2\Omega W_4 - \Omega W_5)_{[m}\Omega_{n]r} + \frac{1}{4}(\Omega W_4)_r \Omega_{mn} + 3\Omega_{[r}{}^k (W_3)_{mn]k} + \\ &+ \frac{1}{2\cdot 4!}\chi_{rmn}{}^{k_1k_2}(W_1)_{k_1k_2} - \frac{1}{4!}\Omega_{[m}{}^k (W_2)_{|k\ell_1\ell_2\ell_3|}\chi^{\ell_1\ell_2\ell_3}{}_{n]r}\,. \end{aligned} \tag{C.9}$$

Now since $\nabla^0 \equiv (\nabla + K^0)$ preserves the SU(5) structure, where $\nabla$ is the Levi-Civita connection, it leaves invariant the spinor $\eta$. Note that the spin connection of $\nabla^0$ is related, in our conventions, to the spin connection of the Levi-Civita connection and the contorsion tensor by,

$$\omega^0{}_m{}^a{}_b = \omega_m{}^a{}_b + K^{0a}{}_{mb}\,, \tag{C.10}$$

where $K^{0a}{}_{mb} \equiv e^a_r e^n_b K^{0r}{}_{mn}$. Using this and (C.8) we see that the spinor $\eta$ solves

$$\begin{aligned} &\Big[\nabla_m + \frac{1}{160}(\Omega W_5 + 5\Omega W_4)_m \Omega_{k_1k_2}\Gamma^{k_1k_2} - \frac{1}{16}(W_4)_k \Gamma_m{}^k + \\ &+ \frac{1}{8}\Omega_m{}^r (W_3)_{rk_1k_2}\Gamma^{k_1k_2} - \frac{1}{394}\chi_{mk_1k_2}{}^{n_1n_2}(W_1)_{n_1n_2}\Gamma^{k_1k_2} + \\ &\qquad + \frac{1}{192}\Omega_m{}^r (W_2)_{r\ell_1\ell_2\ell_3}\chi^{\ell_1\ell_2\ell_3}{}_{k_1k_2}\Gamma^{k_1k_2}\Big]\eta = 0\,. \end{aligned} \tag{C.11}$$

Let us make two further comments about SU(5) structures that are not of direct relevance to the derivations in the text. Firstly, having got explicit expressions for the most general SU(5) preserving connection we can easily see which SU(5)-structures admit a connection with totally antisymmetric torsion. From (C.7) we see that this requires that $T^{(3)}$ and $Q_0^{2,1}$ vanish and also that the structure must satisfy,

$$\begin{aligned} W_2 &= 0 \\ 8W_4 + W_5 &= 0\,. \end{aligned} \tag{C.12}$$

Secondly lets discuss how an SU(5) structure is affected by a conformal transformation of the metric. Consider an SU(5) structure $(g, \Omega, \chi)$ and a transformation $g \to \tilde{g} = e^{2f} g$. The metric $\tilde{g}$ then admits an SU(5) structure $(\tilde{g}, \tilde{\Omega}, \tilde{\chi})$ which is related to the original one by,

$$\begin{aligned} \tilde{\Omega} &= e^{2f}\Omega \\ \tilde{\chi} &= e^{5f}\chi\,. \end{aligned} \tag{C.13}$$

The components $W_i$ then transform as,

$$\begin{aligned} \tilde{W}_1 &= e^{f} W_1 \\ \tilde{W}_2 &= e^{3f} W_2 \\ \tilde{W}_3 &= e^{2f} W_3 \\ \tilde{W}_4 &= W_4 + 8df \\ \tilde{W}_5 &= W_5 - 40df\,. \end{aligned} \tag{C.14}$$

We see that even though the components $W_4$ and $W_5$ transform non-trivially, the combination $W_5 + 5W_4$ is conformally invariant. A simple corollary is that a geometry is conformal to a Calabi-Yau five-fold if and only if it has an SU(5) structure with $W_1 = W_2 = W_3 = 0$ and also has $W_4, W_5$ exact and satisfying $W_5 = -5W_4$.

## D. Some useful identities

We record here some of the identities satisfied by irreps of SU(5) that are useful in deriving the results of table 1 and other formulae. Let $\Lambda^{(p,q)}$ denote a $(p,q)$ form with a subscript of 0 denoting removal of traces, corresponding to an irreducible representation of SU(5) (see (4.7)), then

$$\begin{aligned} *\Lambda_0^{(3,1)} &= -\Lambda_0^{(3,1)} \wedge \Omega \\ *\Lambda_0^{(2,2)} &= \Lambda_0^{(2,2)} \wedge \Omega \\ *\Lambda^{(2,0)} &= \frac{1}{3!}\Lambda^{(2,0)} \wedge \Omega^3 \\ *\Lambda_0^{(1,1)} &= -\frac{1}{3!}\Lambda_0^{(1,1)} \wedge \Omega^3 \\ *\left(\Lambda^{(2,0)} \wedge \Omega\right) &= \frac{1}{2}\Lambda^{(2,0)} \wedge \Omega^2 \end{aligned}$$

$$
\begin{aligned}
*\left(\Lambda_0^{(1,1)} \wedge \Omega\right) &= -\frac{1}{2}\Lambda_0^{(1,1)} \wedge \Omega^2 \\
\Lambda_0^{(1,1)} \wedge \Omega^4 &= 0 \\
\Lambda_0^{(3,1)} \wedge \Omega^2 &= 0 \,. 
\end{aligned}
\tag{D.1}
$$

The following identities are useful in deriving (4.28):

$$
\begin{aligned}
\Gamma_{a_1\ldots a_4}\epsilon &= -\frac{i}{2}\theta_{a_1\ldots a_4 b}\Gamma^b\epsilon^* \\
\Gamma_{a_1\ldots a_3}\epsilon &= \frac{i}{2^2 2!}\theta_{a_1\ldots a_3 b_1 b_2}\Gamma^{b_1 b_2}\epsilon^* \\
\Gamma_{a_1 a_2}\epsilon &= \frac{i}{2^3 3!}\theta_{a_1 a_2 b_1\ldots b_3}\Gamma^{b_1\ldots b_3}\epsilon^* \\
\Gamma_{a_1}\epsilon &= -\frac{i}{2^4 4!}\theta_{a_1 b_1\ldots b_4}\Gamma^{b_1\ldots b_4}\epsilon^* \\
\epsilon &= -\frac{i}{2^5 5!}\theta_{a_1\ldots a_5}\Gamma^{a_1\ldots a_5}\epsilon^* \,.
\end{aligned}
\tag{D.2}
$$

## E. SU(5) structures in eleven dimensions

In deriving the form of the most general geometry admitting a timelike Killing spinor we used the fact that $K$ was a timelike Killing vector and then worked with the SU(5) structure on the $D = 10$ base manifold, $B$, orthogonal to the orbits of $K$. Moreover, our analysis determined the type of SU(5) structure on $B$.

Here we briefly indicate how our analysis also determines the type of $D = 11$ SU(5) structure specified by $(K, \Omega, \Sigma)$ when $K^2 \neq 0$. For example, the fact that $K$ is Killing expresses the vanishing of some components of the corresponding intrinsic torsion. One conceptual advantage of discussing the SU(5) structure in $D = 11$ is that that the rotation parameter $d\omega$ arises as a component of the structure, while from the ten dimensional point of view it is just an arbitrary closed two form.

An $\mathrm{SU}(5) \subset \mathrm{SO}(10,1)$ structure in $D = 11$ can be specified by a one-form $V$, a two-form $J$ and a five-form $\sigma$ such that the vector dual to $V$ is timelike, and the forms satisfy

$$
\begin{aligned}
i_V J &= 0 \\
i_V \sigma &= 0 \\
J \wedge \sigma &= 0 \\
J \wedge i_V * \sigma &= 0 \\
\sigma \wedge i_V * \sigma &= -2^4\frac{J^5}{5!} \,.
\end{aligned}
\tag{E.1}
$$

The one form $V$ allows us to reduce $\mathrm{SO}(10,1) \to \mathrm{SO}(10)$ and $(J, \sigma)$ to further reduce $\mathrm{SO}(10) \to \mathrm{SU}(5)$. We require that the forms defining the structure have constant norm in the eleven dimensional metric (and so are related to rescaled versions of $(K, \Omega, \Sigma)$ as we shall see).

As we discussed in section 3, such structures are classified by the intrinsic torsion $T^0$ which lives in the space $T^* \otimes su(5)^\perp$ where $su(5) \oplus su(5)^\perp = so(10,1)$. The adjoint of

$so(10,1)$ decomposes as $\mathbf{55} \to \mathbf{1} + (\mathbf{5}+\bar{\mathbf{5}}) + (\mathbf{10}+\bar{\mathbf{10}}) + \mathbf{24}$ and so the complement of $su(5)$ is given by $g^{\perp} = \mathbf{1} + (\mathbf{5}+\bar{\mathbf{5}}) + (\mathbf{10}+\bar{\mathbf{10}})$. Noting the following decomposition,

$$
\begin{aligned}
&(\mathbf{1}+\mathbf{5}+\bar{\mathbf{5}}) \otimes (\mathbf{1}+\mathbf{5}+\bar{\mathbf{5}}+\mathbf{10}+\bar{\mathbf{10}}) \to \\
&\quad \to \mathbf{1}+\mathbf{1}'+\mathbf{1}''+(\mathbf{5}+\bar{\mathbf{5}})+(\mathbf{5}+\bar{\mathbf{5}})'+(\mathbf{5}+\bar{\mathbf{5}})''+(\mathbf{10}+\bar{\mathbf{10}})+(\mathbf{10}+\bar{\mathbf{10}})'+ \\
&\qquad +(\mathbf{10}+\bar{\mathbf{10}})''+(\mathbf{15}+\bar{\mathbf{15}})+\mathbf{24}+\mathbf{24}'+(\mathbf{40}+\bar{\mathbf{40}})+(\mathbf{45}+\bar{\mathbf{45}})
\end{aligned}
\tag{E.2}
$$

we see that there are fourteen classes of SU(5) structures in $D=11$, and we can write

$$
T^0 \in \bigoplus_{i=1}^{14} \mathcal{W}_i \,. \tag{E.3}
$$

The intrinsic torsion in each of the modules $\mathcal{W}_i$ can be expressed in terms of the exterior derivatives of $(V,\Omega,\sigma)$. To see this we decompose the exterior derivatives of the forms defining the structure and see which representation appear. Taking into account that they are SU(5) invariant, we find

$$
\begin{aligned}
dV &\to \mathbf{1}+\mathbf{24}+(\mathbf{10}+\bar{\mathbf{10}})+(\mathbf{5}+\bar{\mathbf{5}}) \\
dJ &\to (\mathbf{5}+\bar{\mathbf{5}})+(\mathbf{10}+\bar{\mathbf{10}})+(\mathbf{45}+\bar{\mathbf{45}})+\mathbf{1}+(\mathbf{10}+\bar{\mathbf{10}})'+\mathbf{24} \\
d\sigma &\to (\mathbf{5}+\bar{\mathbf{5}})+(\mathbf{10}+\bar{\mathbf{10}})+(\mathbf{40}+\bar{\mathbf{40}})+\mathbf{1}+\mathbf{1}'+(\mathbf{10}+\bar{\mathbf{10}})'+ \\
&\quad +(\mathbf{15}+\bar{\mathbf{15}})\,.
\end{aligned}
\tag{E.4}
$$

In comparing these to the irreps appearing in the intrinsic torsion, there appears to be a mismatch since four $\mathbf{1}$'s and five $\mathbf{10}+\bar{\mathbf{10}}$'s appear in (E.4) while in (E.2) we have only three $\mathbf{1}$'s and three $\mathbf{10}+\bar{\mathbf{10}}$'s. However this is not so since we can relate the $\mathbf{1}$ of $dJ$ to the $\mathbf{1}+\mathbf{1}'$ of $d\sigma$ by the last condition in (E.1), while the $(\mathbf{10}+\bar{\mathbf{10}})$ of $dJ$ and $d\sigma$ are related by $J\wedge\sigma=0$ and similarly for the $(\mathbf{10}+\bar{\mathbf{10}})'$.

As mentioned, the forms used above to define an SU(5) structure in $D=11$ are not the ones constructed from the Killing spinors. They are related to those, in regions where $K^2=-\Delta^2\neq 0$, by,

$$
\begin{aligned}
K &= \Delta V \\
\Omega &= \Delta J \\
K\wedge\Sigma &= \Delta^2 V\wedge\sigma\,.
\end{aligned}
\tag{E.5}
$$

Given that we know expressions for $dK, d\Omega, d\chi$ we can obtain those for $dV, dJ, d\sigma$ and hence precisely determine the restrictions placed on the $D=11$ $\mathcal{W}_i$. In other words our analysis does indeed determine the $D=11$ SU(5) structure as claimed.

## References


[1] R. L. Bryant, *Pseudo-Riemannian metrics with parallel spinor fields and non-vanishing Ricci tensor*, math.DG/0004073.

[2] J.M. Figueroa-O'Farrill, *Breaking the M-waves*, *Class. and Quant. Grav.* **17** (2000) 2925 [hep-th/9904124].

[3] J. Figueroa-O'Farrill and G. Papadopoulos, *Maximally supersymmetric solutions of ten-dimensional and eleven-dimensional supergravities*, *J. High Energy Phys.* **03** (2003) 048 [hep-th/0211089].

[4] T. Friedrich and S. Ivanov, *Parallel spinors and connections with skew-symmetric torsion in string theory*, math.DG/0102142.

[5] J.P. Gauntlett, N.-w. Kim, D. Martelli and D. Waldram, *Fivebranes wrapped on slag three-cycles and related geometry*, *J. High Energy Phys.* **11** (2001) 018 [hep-th/0110034].

[6] S. Ivanov, *Connection with torsion, parallel spinors and geometry of* Spin(7) *manifolds*, math.DG/0111216.

[7] T. Friedrich and S. Ivanov, *Killing spinor equations in dimension 7 and geometry of integrable G2-manifolds*, math.DG/011220.

[8] J.P. Gauntlett, D. Martelli, S. Pakis and D. Waldram, *G-structures and wrapped NS5-branes*, hep-th/0205050.

[9] J.P. Gauntlett, J.B. Gutowski, C.M. Hull, S. Pakis and H.S. Reall, *All supersymmetric solutions of minimal supergravity in five dimensions*, hep-th/0209114.

[10] S. Gurrieri, J. Louis, A. Micu and D. Waldram, *Mirror symmetry in generalized Calabi-Yau compactifications*, *Nucl. Phys.* **B 654** (2003) 61 [hep-th/0211102].

[11] G.L. Cardoso, G. Curio, G. Dall'Agata, D. Lüst, P. Manousselis and G. Zoupanos, *Non-Kähler string backgrounds and their five torsion classes*, *Nucl. Phys.* **B 652** (2003) 5 [hep-th/0211118].

[12] K.P. Tod, *All metrics admitting supercovariantly constant spinors*, *Phys. Lett.* **B 121** (1983) 241.

[13] J. Gutowski and G. Papadopoulos, *AdS calibrations*, *Phys. Lett.* **B 462** (1999) 81 [hep-th/9902034].

[14] J. Gutowski, G. Papadopoulos and P.K. Townsend, *Supersymmetry and generalized calibrations*, *Phys. Rev.* **D 60** (1999) 106006 [hep-th/9905156].

[15] O. Barwald, N.D. Lambert and P.C. West, *A calibration bound for the M-theory fivebrane*, *Phys. Lett.* **B 463** (1999) 33 [hep-th/9907170].

[16] S. Ivanov and G. Papadopoulos, *A no-go theorem for string warped compactifications*, *Phys. Lett.* **B 497** (2001) 309 [hep-th/0008232].

[17] S. Ivanov and G. Papadopoulos, *Vanishing theorems and string backgrounds*, *Class. and Quant. Grav.* **18** (2001) 1089 [math.DG/0010038].

[18] M. Cvetič, H. Lu and C.N. Pope, *Brane resolution through transgression*, *Nucl. Phys.* **B 600** (2001) 103 [hep-th/0011023].

[19] M. Cvetič, G.W. Gibbons, H. Lu and C.N. Pope, *Ricci-flat metrics, harmonic forms and brane resolutions*, *Commun. Math. Phys.* **232** (2003) 457 [hep-th/0012011].

[20] K. Becker and M. Becker, *M-theory on eight-manifolds*, *Nucl. Phys.* **B 477** (1996) 155 [hep-th/9605053].

[21] M.J. Duff, J.M. Evans, R.R. Khuri, J.X. Lu and R. Minasian, *The octonionic membrane*, *Phys. Lett.* **B 412** (1997) 281 [hep-th/9706124].

JHEP04(2003)039

[22] S.W. Hawking and M.M. Taylor-Robinson, *Bulk charges in eleven dimensions*, *Phys. Rev.* **D 58** (1998) 025006 [hep-th/9711042].

[23] C. Vafa and E. Witten, *A one loop test of string duality*, *Nucl. Phys.* **B 447** (1995) 261 [hep-th/9505053].

[24] M.J. Duff, J.T. Liu and R. Minasian, *Eleven-dimensional origin of string/string duality: a one-loop test*, *Nucl. Phys.* **B 452** (1995) 261 [hep-th/9506126].

[25] H.S. Reall, *Higher dimensional black holes and supersymmetry*, hep-th/0211290.

[26] E. Bergshoeff, E. Sezgin and P.K. Townsend, *Supermembranes and eleven-dimensional supergravity*, *Phys. Lett.* **B 189** (1987) 75.

[27] E. Bergshoeff, E. Sezgin and P.K. Townsend, *Properties of the eleven-dimensional super membrane theory*, *Ann. Phys. (NY)* **185** (1988) 330.

[28] S. Salamon, *Riemannian geometry and holonomy groups*, Pitman Reseacrh Notes in Mathematics 201, Longman, 1989.

[29] S. Salamon, *Almost parallel structures*, math.DG/0107146.

[30] S. Chiossi and S. Salamon, *The intrinsic torsion of* SU(3) *and G2 structures*, math.DG/0202282.

[31] A. Gray and L. Hervella, *The sixteen classes of almost hermitian manifolds*, *Ann. Mat. Pura e Appl.* **282** (1980) 1.

[32] J.P. Gauntlett, N. Kim and D. Waldram, *M-fivebranes wrapped on supersymmetric cycles*, *Phys. Rev.* **D 63** (2001) 126001 [hep-th/0012195].

[33] B. Biran, F. Englert, B. de Wit and H. Nicolai, *Gauged $N = 8$ supergravity and its breaking from spontaneous compactification*, *Phys. Lett.* **B 124** (1983) 45.